Furstenberg's structure theorem via CHART groups

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Abstract

We give an almost self-contained group theoretic proof of Furstenberg's structure theorem as generalized by Ellis: Each minimal compact distal flow is the result of a transfinite sequence of equicontinuous extensions, and their limits, starting from a flow consisting of a singleton. The groups that we use are CHART groups, and their basic properties are recalled at the beginning of this paper.

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0 Introduction

The "CHART" in the title stands for "compact Hausdorff admissible right topological", that is, a CHART group (G, τ) is a group G, with a compact Hausdorff topology τ, such that right multiplication $x \mapsto xy$ is continuous for each $y \in G$ and left multiplication $x \mapsto yx$ is continuous for each y belonging to a dense subset of G. The purpose of the present note is to give an almost self-contained proof of the Furstenberg structure theorem, as generalized by Ellis, using CHART groups.

In 1963, Furstenberg published his ground breaking study of distal flows [4] in which he proved, in part, that each compact metric minimal distal flow is the result of a transfinite sequence of isometric extensions, and their limits, starting from a trivial flow consisting of a singleton. In 1978 Ellis [3] generalized this structure theorem to arbitrary compact distal flows, where equicontinuous extensions replaced isometric extensions. The second author of the present note published in 1972 [8] a proof of Furstenberg's structure theorem using CHART groups and their σ-topology. This approach is used again in the present note but without metric or countability assumptions. The initial part of [8], which has to do with properties of the σ-topology, will not be repeated in full here. Instead, the relevant facts are quoted, and for their proofs, references to [8] and the more recent paper [9] are given. Apart from these, the paper is selfcontained. In particular, no additional techniques from topological dynamics are used.

This note is structured as follows. There are four sections. Section 0 is the introduction. Section 1 contains preliminary material such as definitions and results from earlier papers to be used without proof. In Section 2, the notions from topological dynamics, such as flow maps and equicontinuous extensions are interpreted group theoretically, and finally a proof of the Furstenberg-Ellis structure theorem is given in Section 3. One by-product of the present approach is that the proof, due to Milnes and Pym [6], of the existence of a Haar measure for separable CHART groups, now works for arbitrary CHART groups. Thus, CHART groups are the next best thing after compact Hausdorff topological groups.

1 Preliminaries

1.1 Flows

An *action* of a semigroup S on a set X is a function $S \times X \to X$ (denoted by $(s, x) \mapsto s \cdot x$) such that $s \cdot (t \cdot x) = (st) \cdot x$ for all $s, t \in S$ and $x \in X$. In the case when X is a topological space and the map $x \mapsto s \cdot x$ is continuous, for each $s \in S$, the action is called a *continuous action* or alternatively that S *acts continuously on* the topological space X. A *flow* is a pair (S, X), where X is a Hausdorff space and S is a semigroup acting continuously on X. A *compact flow* is a flow (S, X) in which X is a compact Hausdorff space. A *flow map* $f : (S, X) \to (S, Y)$ of (S, X) into (S, Y) is a continuous map $f : X \to Y$ such that $f(s \cdot x) = s \cdot f(x)$ for all $s \in S$ and $x \in X$. If f is an inclusion map $X \subset Y$, then (S, X) is called a *subflow* of (S, Y). If the map f is a quotient map, then (S, Y) is said to be a *factor* of (S, X) and (S, X) is said to be an *extension* of (S, Y). If f is a homeomorphism of X onto Y, then f is a *flow isomorphism* and the flows (S, X) and (S, Y) are said to be *isomorphic*. A flow is said to be *minimal* if it does not have any proper subflow. Clearly if (S, X) is a compact flow, then (S, X) is minimal if and only if, for each $x \in X$, $S \cdot x = \{s \cdot x : s \in S\}$ is dense in X.

1.2 Compact flows

Let (S, X) be a compact flow and define the map $\eta : S \to X^X$ by $\eta(s)(x) = s \cdot x$ for each $s \in S$ and $x \in X$. Provide X^X with the product topology τ, *i.e.* the topology of pointwise convergence. Then (X^X, τ) is a compact Hausdorff space with a semigroup structure given by the composition of maps. The mapping $\eta : S \to X^X$ is a homomorphism of semigroups and the closure $\Sigma = \overline{\eta(S)}^{\tau}$ in X^X is a compact subsemigroup of X^X. This Σ is called the *enveloping semigroup* of the flow (S, X). The enveloping semigroup Σ has the following properties:

(i) The map $\alpha \mapsto \alpha\beta$ is continuous for each $\beta \in \Sigma$, *i.e.* the right multiplication is continuous.

(ii) The map $\beta \mapsto \eta(s)\beta$ is continuous for each $s \in S$, *i.e.* left multiplication by members of $\eta(S)$ is continuous.

The following theorem due to Ellis [1] is basic in the present note. For a proof, see [9, Proposition 3.1].

1.1 Theorem (Ellis)
Let Σ be the enveloping semigroup of a compact flow (S, X). Then the following statements are equivalent.

(a) Each member of Σ is one-to-one.

(b) Each member of Σ is onto.

(c) Σ is a group with the identity element $id_X : X \to X$.

A compact flow (S, X) is said to be *distal* if one (hence all) of the conditions of the theorem above is satisfied. Clearly condition (a) is equivalent to the statement: if $x, y \in X$ and for some net $\{s_\gamma\}$ in S, $\lim_\gamma s_\gamma \cdot x = \lim_\gamma s_\gamma \cdot y$, then $x = y$.

The enveloping semigroup of a distal flow is called the *Ellis group* of the flow. The Ellis group is a compact group satisfying the continuity conditions (i) and (ii) above.

1.3 Right topological groups

Abstracting the Ellis groups described above, we define a *right topological group* to be a pair (G,τ), where G is a group and τ is a topology (not necessarily Hausdorff) on G such that right multiplication is continuous, *i.e.* for each $y \in G$ the map $x \mapsto xy$ is continuous. Similarly a *left topological group* is defined by replacing the right multiplication by left multiplication $x \mapsto yx$. If (G,τ) is both a right and left topological group then it is called a *semitopological group*.

Let (G,τ) be a right topological group and let H be a (not necessarily closed) subgroup. Let (H,τ) be the space H with the relativization to the topology τ. Clearly (H,τ) is again a right topological group. We let $(G/H,\tau)$ be the space $\{xH : x \in G\}$ of left cosets of H in G with the quotient topology induced from (G,τ) by the quotient map $\pi : G \to G/H$ given by $x \mapsto xH$. We remark that π is open since, if U is an open subset of G, then $\pi^{-1}\pi(U) = UH = \bigcup\{Ux : x \in H\}$ which is open since right multiplications are homeomorphisms. The following Lemma justifies our notation for relative and quotient topologies [8, p.197], [9, Lemma 4.2].

1.2 Lemma
Let L, H be subgroups of a right topological group (G,τ) such that $H \subset L$. Then regarding L/H as a subset of G/H, the relative topology induced on L/H from $(G/H,\tau)$ is identical with the quotient topology induced from (L,τ) by the quotient map $L \to L/H$.

1.4 The σ-topology

Let (G,τ) be a right topological group and let $\varphi : G \times G \to G$ be the map defined by $\varphi(x,y) = x^{-1}y$. Then the quotient topology on G induced from $(G \times G, \tau \times \tau)$ by the map φ is called the $\sigma(G,\tau)$*-topology* (or σ*-topology*, when there is no confusion). The following is a summary of the properties of the σ-topology [8, Theorem 1.1], [9, Lemma 4.3].

1.3 Lemma
Let (G,τ) be a right topological group and let σ be its σ- topology. Then

(a) (G,σ) is a semitopological group and the inverse map $x \mapsto x^{-1}$ is σ-continuous.

(b) $\sigma \subset \tau$.

(c) A subgroup H of G is σ-closed if and only if $(G/H,\tau)$ is Hausdorff.

Let (G,τ) be a right topological group and let $\Lambda(G,\tau)$ (or simply $\Lambda(G)$ when no confusion is possible) be the set of all $x \in G$ such that the map $y \mapsto x \cdot y$ is τ-continuous. It is easy to check that when (G,τ) is compact and Hausdorff, $\Lambda(G,\tau)$ is a subgroup of G. If $\Lambda(G)$ is τ-dense in G, then (G,τ) is said to be *admissible*. For instance, if (S,X) is a distal flow, then its Ellis group (Σ,τ) is admissible, because as seen above $\eta(S) \subset \Lambda(G)$ and $\eta(S)$ is τ-dense in Σ.

1.4 Lemma
Let (G,τ) be a compact right topological group and let K be a σ-closed subgroup of G. Let S be a semigroup and let $\eta : S \to \Lambda(G)$ be a semigroup homomorphism. If we define a continuous action of S on the compact Hausdorff space $(G/K,\tau)$ by $s \cdot gK = \eta(s)gK$, then the flow $(S,(G/K,\tau))$ is distal. If $\eta(S)$ is dense in G, then this flow is also minimal.

Proof. Suppose g_1 and g_2 are elements of G such that for some net $\{s_\gamma\}$ in S, $\lim_\gamma s_\gamma \cdot g_1K = \lim_\gamma s_\gamma \cdot g_2K$ or $\lim_\gamma \eta(s_\gamma)g_1K = \lim_\gamma \eta(s_\gamma)g_2K$. Let $k \in G$ be a τ-cluster point of the net $\{\eta(s_\gamma)\}$. Then $kg_1K = kg_1K$ or $g_1K = g_2K$. This proves that $(S, G/K)$ is distal. If $\eta(S)$ is dense in G, then for each $g \in G$, $\eta(S)g$ is dense in G since (G, τ) is a right topological group. Hence the flow $(S, G/K)$ is minimal.

The proof of the next proposition is found in [8, Theorem 1.2] or [9, Proposition 4.4].

1.5 Proposition
Let (G, τ) be an admissible right topological group. Then:

(a) The quotient map $\varphi : (G \times G, \tau \times \tau) \to (G, \sigma)$ is open.

(b) If $\mathcal{U}$ is the family of all τ-open neighborhoods of e in G then $\{U^{-1}U : U \in \mathcal{U}\}$ is a base of open neighborhoods of e in (G, σ).

The following proposition is proved in [9, Proposition 4.5].

1.6 Proposition
Let (G, τ) be an admissible right topological group, let $\mathcal{U}$ be the family of all τ-open neighborhoods of e in (G, τ) and let $N = \bigcap\{\overline{U}^\tau : U \in \mathcal{U}\}$. Then:

(a) $N = \bigcap\{U^{-1}U : U \in \mathcal{U}\}$ and N is a σ-closed (hence τ-closed) normal subgroup of G.

(b) For $x \in G$, $x \in N$ if and only if there is a net which τ-converges simultaneously to both e and x.

(c) $(G/N, \tau)$ is a compact Hausdorff admissible right topological group. If (G, τ) is a semitopological group, then so is $(G/N, \tau)$.

1.5 CHART groups

Let (G, τ) be a CHART group, *i.e.* a compact Hausdorff admissible right topological group and let σ be its σ-topology. Let L be a σ-closed subgroup of G. Recall that (L, σ) is semitopological group, hence it is an admissible group. Then by $N(L, \sigma)$ (or $N(L)$ when there is no confusion) we denote the intersection of all σ-closed σ-neighborhoods of e in L. Then by Proposition 1.6 (a), $N(L)$ is a $\sigma(L, \sigma)$-closed normal subgroup of L. Hence by Lemma 1.3(c), $(L/N(L), \sigma)$ is a compact Hausdorff semitopological group. Hence by Ellis' theorem [1], $(L/N(L), \sigma)$ is a compact Hausdorff topological group, and since the topology of $(L/N(L), \tau)$ is stronger than σ, $(L/N(L), \sigma) = (L/N(L), \tau)$. We can make the definition of $N(L)$ a little more explicit. As above, let $\mathcal{U}$ be the family of all open neighborhoods of e in (G, τ). Then by Proposition 1.5(b), $\{U^{-1}U \cap L : U \in \mathcal{U}\}$ is a base of open neighborhoods of e in (L, σ). Therefore by Proposition 1.6(a), $N(L) = \bigcap\{(U^{-1}U \cap L)^{-1}(U^{-1}U \cap L) : U \in \mathcal{U}\}$. This formula and part (a) of the following proposition are observations due to Milnes and Pym [6]. Part(b) is proved in [8, Proposition 2.1] and parts (a) and (b) are proved in [9, Proposition 4.6].

1.7 Proposition
Let (G, τ) be a CHART group. Then using the notation given above, we have the following properties:

(a) If L is a normal subgroup of G, then so is $N(L)$.

(b) If $m : (G/N(L), \tau) \times (L/N(L), \tau) \to (G/N(L), \tau)$ is defined by $(xN(L), yN(L)) \mapsto xyN(L)$ for each $x \in G$ and $y \in L$, then m is well-defined and continuous.

Remark Let (G, τ) be a CHART group and let $\mathcal{U}$ be the family of all open neighborhoods of $e \in G$. Then if $g \in G$ and $g \neq e$, then, for some $U \in \mathcal{U}$, $U \cap Ug = \emptyset$ or $g \notin U^{-1}U$, since τ is Hausdorff. Therefore $\bigcap\{U^{-1}U : U \in \mathcal{U}\} = \{e\}$. Suppose (G, τ) satisfies the first countability axiom. Then $\mathcal{U}$ admits a countable base $\{U_n : n \in \mathbb{N}\}$, and hence $\bigcap\{U_n^{-1}U_n : n \in \mathbb{N}\} = \{e\}$, *i.e.* $\{e\}$ is a G_δ-point in (G, σ). Let $\varphi : (G \times G, \tau \times \tau) \to (G, \sigma)$ be the quotient map as in the Subsection 1.4. Then the diagonal $\Delta_G = \varphi^{-1}(\{e\})$ is a $\tau \times \tau$-G_δ subset of $G \times G$. Hence (G, τ) is metrizable (see [5, Exercise 4.2B]), and by [8, Theorem 2.1], (G, τ) is a topological group. Hence *a CHART group which satisfies the first countability axiom is a metrizable topological group.*

2 Equicontinuous flow maps

2.1 Representation of flow maps between minimal distal flows

Let (S, X) and (S, Y) be compact flows and let $\pi : (S, X) \to (S, Y)$ be a quotient (*i.e.* continuous and onto) flow map. Let Σ_X, Σ_Y be the enveloping semigroups of (S, X) and (S, Y) respectively. Then for each $\lambda \in \Sigma_X$ there is a $\mu \in \Sigma_Y$ such that $\pi\lambda = \mu\pi$. To see this let $\{s_\gamma\}$ be a net in S such that $s_\gamma \cdot x \to \lambda(x)$ for each $x \in X$. By taking a subnet, we may assume that, for some $\mu \in \Sigma_Y$, $s_\gamma \cdot y \to \mu(y)$ for each $y \in Y$. Then $\pi(s_\gamma \cdot x) \to \pi(\lambda(x))$ and $\pi(s_\gamma \cdot x) = s_\gamma \cdot \pi(x) \to \mu(\pi(x))$. It follows $\pi(\lambda(x)) = \mu\pi(x)$ for each $x \in X$. Hence $\pi\lambda = \mu\pi$. Since π is onto, this equation shows that μ is determined uniquely by λ alone. Denote $\mu = \pi_*(\lambda)$. Then we have the following relationship

$$\pi\lambda = \pi_*(\lambda)\pi. \tag{2.1}$$

A straightforward verification shows that $\pi_* : \Sigma_X \to \Sigma_Y$ is a continuous onto homomorphism of semigroups.

Let us further assume that the flow (S, X) is a minimal distal flow. Then (S, Y) is also minimal, since, if $y \in Y$ then $y = \pi(x)$ for some $x \in X$ and $S \cdot x$ is dense in X and so $\pi(S \cdot x) = S \cdot y$ is dense in Y. Since (S, K) is distal each $\lambda \in \Sigma_X$ is an onto map by Theorem 1.1(b), and therefore, $\pi_*(\lambda)$ is onto by equation (2.1) above. Using $\pi_*(\Sigma_X) = \Sigma_Y$, we see that each element of Σ_Y is onto. Hence the flow (S, Y) is distal by the same theorem. Let G_X, G_Y be the Ellis groups of $(S, X), (S, Y)$ with identity elements e_X, e_Y respectively. Then $\pi_* : G_X \to G_Y$ satisfies $\pi_*(e_X) = e_Y$ and π_* is a continuous onto homomorphism of groups.

Let x_0 be a fixed base point of X and let $y_0 = \pi(x_0)$. Define the maps $h_X : G_X \to X$ and $h_Y : G_Y \to Y$ by $h_X(g) = g(x_0)$ and $h_Y(g') = g'(y_0)$ for $g \in G_X, g' \in G_Y$. By the minimality of the flows, both h_X and h_Y are onto and continuous. Let $K = h_X^{-1}(x_0)$. Then K is closed in G_X and it can be checked directly that it is a subgroup of G_X. Furthermore, for $g_1, g_2 \in G_X$, $h_X(g_1) = h_X(g_2)$ if and only if $g_1^{-1}g_2 \in K$. This means that the map $\alpha : G_X/K \to X$ defined by $gK \mapsto g(x_0)$ for $g \in G_X$ is a homeomorphism. Since

X is Hausdorff, K is σ-closed in G_X, by Lemma 1.3(c). Let $h = \pi h_X : G_X \to Y$. Then for $g \in G_X$, by using (2.1), $h(g) = \pi(g(x_0)) = \pi_*(g)\pi(x_0) = \pi_*(g)(y_0) = h_Y \pi_*(g)$, *i.e.*

$$\pi h_X = h = h_Y \pi_*. \tag{2.2}$$

Now let $L = h^{-1}(y_0) = \pi_*^{-1}(H)$, where $H = h_Y^{-1}(y_0)$. Then since H is a closed subgroup of G_Y and π_* is a continuous homomorphism, L is a closed subgroup of G_X. For $g_1, g_2 \in G_X,\ h(g_1) = h(g_2)$ if and only if $\pi_*(g_1^{-1}g_2) = \pi_*(g_1)^{-1}\pi_*(g_2) \in H$ and this is the case if and only if $g_1^{-1}g_2 \in L$. So as before, the map $\beta :\ G_X/L \to Y$ given by $gL \mapsto h(g) = (\pi_* g)(y_0)$ is a homeomorphism and L is σ-closed in G_X. From formula (2.2), it is clear that $K \subset L$. Finally, we wish to transfer the flow map $\pi : X \to Y$ to a map $G_X/K \to G_X/L$. Let $g \in G_X$. Then using (2.1), $\pi\alpha(gK) = \pi(g(x_0)) = \pi_* g(y_0) = h(g) = \beta(gL)$. Let $\rho : G_X/K \to G_X/L$ be the map defined by $\rho(gK) = gL$ for $g \in G$. Then from above, $\pi\alpha = \beta\rho$.

Lastly, we show that the various maps introduced above are flow maps. Suppose that the action of S on X is given by the semigroup homomorphism $\eta : S \to \Lambda(G_X)$ (see Subsection 1.2), and write G for G_X and η for η_X. Then the continuous actions of S on G, G/K and G/L are given by $s \cdot g = \eta(s)g$, $s \cdot (gK) = (\eta(s)g)K$ and $s \cdot (gL) = (\eta(s)g)L$ for each $s \in S$, $g \in G$. We first check that $h_X : G \to X$ is a flow map. So let $s \in S$ and $g \in G$. Then $h_X(s \cdot g) = h_0(\eta(s)g) = \eta(s)(g(x_0)) = \eta(s)h_X(g) = s \cdot h_X(g)$. Hence h_X is a flow map. Recall that the map $\alpha : G/K \to X$ is defined by $\alpha(gK) = h_X(g) = g(x_0)$. Therefore for $s \in S$ and $g \in G$, $s \cdot \alpha(gK) = s \cdot h_X(g) = h_X(s \cdot g) = \alpha((\eta(s)g)K) = \alpha(s \cdot (gK))$, *i.e.* α is a flow map. Since h_X and π are flow maps, so is $h = \pi h_X$ and consequently the map β is a flow map. Clearly the map ρ is a flow map.

We summarize the discussion above as follows. The converse part of (i) follows from Lemma 1.4. Part (ii) states that an arbitrary factor of the minimal distal flow (X, S) is represented by the quotient of the Ellis group (G, τ) by a σ-closed subgroup.

2.1 Proposition
Let (S, X) be a minimal distal flow and let (G, τ) be its Ellis group. Then

- *(i) There exists a σ-closed subgroup K of G and a semigroup homomorphism $\eta : S \to \Lambda(G)$ with dense range such that the flow (S, X) is isomorphic to the flow $(S, (G/K, \tau))$ where the action of S on G/K is as given in Lemma 1.4. Conversely, given a CHART group (G, τ), a σ-closed subgroup K and a semigroup homomorphism of a semigroup S onto a dense subsemigroup of $\Lambda(G)$, the flow $(S, (G/K, \tau))$, as defined in Lemma 1.4, is a minimal distal flow.*
- *(ii) Let (S, Y) be a factor of (S, X) by a quotient flow map $\pi : X \to Y$. Then (S, Y) is also a minimal distal flow and there exist σ-closed subgroups K and L of (G, τ) with $K \subset L$ and a semigroup homeomorphism $\eta : S \to \Lambda(G)$ with dense range such that:*
  - *(a) there exist flow isomorphisms $\alpha : (S, G/K) \to (S, X)$ and $\beta : (S, G/L) \to (S, Y)$ where the actions of S on G/K and G/L are as in Lemma 1.4;*
  - *(b) if $\rho : (S, G/K) \to (S, G/L)$ is the map $gK \mapsto gL$ $(g \in G)$, then ρ is a flow map and $\pi\alpha = \beta\rho$.*

2.2 Equicontinuous flow maps

Let (S, X) be a minimal distal flow with Ellis group (G, τ) and let σ be its σ-topology. *To simplify our notation, we will assume that S is a dense subsemigroup of $\Lambda(G)$ rather than a homomorphic image of S.* Since X is compact and Hausdorff, it has a unique uniform structure $\mathcal{U}_X$ which is the family of all open neighborhood of the diagonal $\Delta_X = \{(x, x) : x \in X\}$ in $X \times X$. A quotient flow map $\pi : (S, X) \to (S, Y)$ is said to *equicontinuous* if for each uniformity $U \in \mathcal{U}_X$ there exists a uniformity $V \in \mathcal{U}_X$ such that if $(x, y) \in V$ and $\pi(x) = \pi(y)$, then $s(x, y) \in U$ for each $s \in S$ where $s(x, y) = (sx, sy)$. Let $M = \{(x, y) \in X \times X : \pi(x) = \pi(y)\}$. Since the map $t \mapsto (tx, ty)$ is continuous for each $(x, y) \in X \times X$, we may state the last part of the definition as $\overline{G(M \cap V)} \subset \overline{U}$. Now, by the definition of M, it is clear that $sM \subset M$ for each $s \in S$. Hence by the continuity noted above, $gM \subset M$ for each $g \in G$ and consequently $GM = M$ since G is a group. It follows that $G(M \cap V) = M \cap GV$. Since $U \in \mathcal{U}_X$ is arbitrary, we conclude that π is equicontinuous if and only if

$$\bigcap\{\overline{M \cap GV)} : V \in \mathcal{U}_X\} = \Delta_X. \tag{2.3}$$

Let (G, τ) be a CHART group and let K and L be σ-closed subgroups of G with $K \subset L$. Let $X = (G/K, \tau)$ and $Y = (G/L, \tau)$ and let S be a dense subsemigroup of $\Lambda(G)$. Then X, Y are compact Hausdorff spaces because K, L are σ-closed subgroups by Lemma 1.3(c). Let S act on X and Y by the maps $(s, gK) \mapsto sgK$ and $(s, gL) \mapsto sgL$. Then (S, X) and (S, Y) are minimal distal flows by Proposition 2.1(i). Define the map $\pi : X \mapsto Y$ by $\pi(gK) = gL$. Then $\pi : (S, X) \to (S, Y)$ is a flow map.

2.2 Theorem
Using the notation given above, the following statements are equivalent:

- *(a) The map π is equicontinuous.*
- *(b) The space $((L/K), \sigma)$ is Hausdorff.*
- *(c) $N(L) \subset K$, where $N(L)$ is as in Proposition 1.7.*

Proof.

$(a) \Rightarrow (c)$. Suppose that $x \in N(L)$ and let $p : G \to G/K = X$ be the quotient map $g \mapsto gK$. We must prove that $x \in K$, *i.e.* $p(x) = p(e)$ or show that $(p(e), p(x)) \in \Delta_{G/K} = \Delta_X$. Since π is equicontinuous by (a), using (2.3) it is sufficient to show that, for each $V \in \mathcal{U}_X$, $(p(e), p(x)) \in \overline{(M \cap GV)}$ holds.

To this end, let W be an arbitrary neighborhood of $(p(e), p(x))$ in $X \times X$. Next, choose a neighborhood U of e in (G, τ) such that $(p \times p)(U \times Ux) \subset W$ and $(p \times p)(U \times U) \subset V$. Here recall that the quotient map $p : G \to X$ is open and hence the map $p \times p : G \times G \to G/K \times G/K$ is also open.

Now, since $x \in N(L)$, $x \in (U^{-1}U \cap L)^{-1}(U^{-1}U \cap L)$ or equivalently, $(U^{-1}U \cap L)x \cap (U^{-1}U \cap L) \neq \emptyset$. Choose $z \in (U^{-1}U \cap L)x \cap (U^{-1}U \cap L)$. Then there are $u, v \in U$ such that $uz \in Ux$ and $vz \in U$. (Note $z \in L$.) Consequently $(u, uz) \in U \times Ux$ and $(u, uz) = g(v, vz)$ for $g = uv^{-1}$. It follows that $(p \times p)(u, uz) \in (p \times p)(U \times Ux) \subset W$ and $(p \times p)(u, uz) = g(p \times p)(v, vz) \in g(p \times p)(U \times U) \subset gV$. On the other hand, since $z \in L$, $(\pi \times \pi)(p \times p)(u, uz) = (uL, uzL) = (uL, uL)$, *i.e.* $(p \times p)(u, uz) \in M$. Combining the inclusions noted above, we obtain that $(p \times p)(u, uz) \in (M \cap gV) \subset (M \cap GV)$. Since $(p \times p)(u, uz) \in W$, we see that $W \cap (M \cap GV) \neq \emptyset$. Since W is an arbitrary neighborhood of $(p(e), p(x))$ in $X \times X$, $(p(e), p(x)) \in \overline{(M \cap GV)}$ as desired.

$(c) \Rightarrow (b)$. As noted at the beginning of Subsection 1.3, the quotient maps $L \to L/N(L)$ and $L \to L/K$ are continuous and open in τ and σ topologies. Hence the natural map $L/N(L) \to L/K$ is also continuous and open in τ and σ. However since the two topologies agree on $L/N(L)$ (see Subsection 1.5), they also agree on L/K. Since K is σ-closed, $(L/K, \tau)$ is Hausdorff (see Lemma 1.3(c)). Hence $(L/K, \sigma)$ is Hausdorff.

$(b) \Rightarrow (a)$. As before let $\varphi : (G \times G, \tau \times \tau) \to (G, \sigma)$ be the map $(x, y) \mapsto x^{-1}y$. Then by the definition of σ-topology (Subsection 1.4) φ is continuous, and by Proposition 1.5(a) it is also open. Let $p : G \to G/K$ be the quotient map. Let $\mathcal{V}$ denote the family of all open neighborhoods of e in $(G/K, \sigma)$. Then $\{V \cap (L/K) : V \in \mathcal{V}\}$ is a base for the neighborhoods of $p(e)$ in $(L/K, \sigma)$, hence $\{\overline{V \cap (L/K)}^{\sigma} : V \in \mathcal{V}\}$ is a base for the family of closed neighborhoods of $p(e)$ in $(L/K, \sigma)$. Since $(L/K, \sigma)$ is Hausforff by (b),

$$\{p(e)\} = \bigcap \{\overline{V \cap (L/K)}^{\sigma} : V \in \mathcal{V}\}.$$

By applying p^{-1} to the both sides of the above, we obtain:

$$K \subset \bigcap \{\overline{p^{-1}(V) \cap L}^{\sigma} : V \in \mathcal{V}\} \subset \bigcap \{p^{-1}(\overline{(V \cap L/K)}^{\sigma}) : V \in \mathcal{V}\} = K.$$

Therefore $K = \bigcap \{\overline{p^{-1}(V) \cap L}^{\sigma} : V \in \mathcal{V}\}$. Similarly apply φ^{-1} to the preceding to obtain:

$$\varphi^{-1}(K) = \bigcap \{\overline{(p\varphi)^{-1}(V) \cap \varphi^{-1}(L)}^{\tau} : V \in \mathcal{V}\}.$$

Since $(p \times p)^{-1}(\Delta_X) = \varphi^{-1}(K)$, if U is an open neighborhood of the diagonal $\Delta_X \subset X \times X = (G/K \times G/K, \tau \times \tau)$, then $(p \times p)^{-1}(U)$ is an $\tau \times \tau$- neighborhood of $\varphi^{-1}(K)$. Hence for some $V \in \mathcal{V}$,

$$\varphi^{-1}(L) \cap (p\varphi)^{-1}(V) \subset (p \times p)^{-1}(U). \tag{2.4}$$

Note $(p\varphi)^{-1}(V)$ is an open neighborhood of $\Delta_G \subset (G \times G, \tau \times \tau)$. Since $p \times p : (G \times G, \tau \times \tau) \to (G/K \times G/K, \tau \times \tau) = (X \times X)$ is open, the set $W = (p \times p)((p\varphi)^{-1}(V))$ is an open neighborhood of Δ_X. Since for each subset A of G, $\varphi^{-1}(A) = G\varphi^{-1}(A)$ and since the map $p \times p : G \times G \to G/K \times G/K$ commutes with the action of G, we have $GW = W$. Also note that $\varphi^{-1}(L) = (p \times p)^{-1}(M)$. Hence by applying $p \times p$ to (2.4), we obtain $M \cap GW = W \cap M \subset U$. [Here we used an easily provable fact: if $f : A \to B$ and C and D are subsets of A and B respectively, then $f(C \cap f^{-1}(D)) = f(C) \cap D$.] It follows that $\overline{M \cap GW} \subset \overline{U}$. Since U is an arbitrary open neighborhood of Δ_X, $\bigcap \{\overline{M \cap GW)} : W \in \mathcal{U}_X\} = \Delta_X$ (see (2.3)). Therefore $\pi : X \to Y$ is equicontinuous.

3 The structure theorems

3.1 The basic theorem

Let (G, τ) be a CHART group and define inductively $\{L_\xi\}$ as follows: $L_0 = G, L_1 = N(L_0), L_2 = N(L_1), \cdots$ etc, where, as in Section 1 and 2, for a subgroup L of G, $N(L)$ denotes the intersection of all closed neighborhoods of e in (L, σ). Then each L_ξ is a σ-closed normal subgroup of G, by Propositions 1.6, 1.7 and induction. Furthermore by Theorem 2.2 the map $G/L_{\xi+1} \to G/L_\xi$ is an equicontinuous flow map, *i.e.* $G/L_{\xi+1}$ is an equicontinuous extension of G/L_ξ. However, this inductive process stops as soon as $L_{\xi+1} = L_\xi$. The next theorem guarantees that this is not the case so long as $G/L_\xi \neq \{e\}$.

We need the following lemma due to Furstenberg [4]. Since its proof is very simple, we give it here.

3.1 Lemma
Let (G, τ) be a compact right topological group and let ω be a topology on G weaker than τ such that (G, ω) is also a right topological group. If U is an open dense subset of (G, ω), then U also τ-dense in G.

Proof. Let $C = G \setminus U$. Then by the hypothesis, C is a ω-closed (hence τ-closed) nowhere dense subset of G. If U is not τ-dense in G, then C contains a nonempty τ-open subset. By the compactness of (G, τ) there exists a finite subset F of G such that $G = \bigcup\{Cg : g \in F\}$. Now each Cg is nowhere dense in (G, ω) since each right multiplication is a homeomorphism there. This contradicts the fact that a nonempty topological space can never be the union of a finite number of nowhere dense subsets.

3.2 Lemma
Let (G, τ) be a CHART group and let $\Lambda = \Lambda(G)$. If A and B are nonempty open subsets of (G, τ), then $A^{-1}B = (A \cap \Lambda)^{-1}B$.

Proof. Let $x \in A^{-1}B$. Then for some $a \in A$, $ax \in B$. Since B is open and $A \cap \Lambda$ is dense in A, there is a $c \in A \cap \Lambda$ such that $cx \in B$. Hence $x \in c^{-1}B \subset (A \cap \Lambda)^{-1}B$. Hence $A^{-1}B \subset (A \cap \Lambda)^{-1}B$. The reverse inclusion is obvious.

3.3 Lemma
Let (G, τ) be a compact Hausdorff right topological group. If S is a subsemigroup of $\Lambda(G)$ then the closure $\overline{S}$ of S in (G, τ) is a subgroup of G.

Proof. The semigroup S acts on G continuously by the map $(s, x) \mapsto sx$ for $(s, x) \in S \times G$. Since (G, τ) is a compact Hausdorff right topological group, the enveloping semigroup of the flow (S, G) can be identified with $\overline{S}$, where each $t \in \overline{S}$ acts on G by the left multiplication $x \mapsto tx \quad (x \in G)$. Since G is a group each left multiplication is onto. Hence by Theorem 1.1, $\overline{S}$ is a subgroup of G.

3.4 Theorem
Let (G, τ) be a CHART group and let σ denote its σ-topology. Suppose that K and L are σ-closed subgroups of G such that $K \subset L$ and $K \neq L$. Let $H = N(L)K$, where $N(L)$ is the intersection of all closed neighborhoods of e in (L, σ) (see Subsection 1.5). Then H is a σ-closed subgroup of L with $H \neq L$ and $K \subset H$.

Proof. As noted in Subsection 1.5, the σ and τ topologies agree on $L/N(L)$. Since the quotient map $q : (L, \tau) \to (L/N(L), \tau)$ is continuous, $q(K)$ is τ-closed and hence σ-closed. Since q is also $\sigma - \sigma$ continuous $q^{-1}q(K) = KN(L)$ is σ-closed in L hence in G. As seen in the remark prior to Proposition 1.7, $N(L)$ is normal in L. Since $K \subset L$, $H = KN(L) = N(L)K$ is a subgroup of L (hence of G). Clearly $H \subset L$. So it remains to prove that $H \neq L$.

Let $\mathcal{U}$ denote the family of all open neighborhood of e in (G,τ) and let $\mathcal{V} = \{U^{-1}U : U \in \mathcal{U}\}$. Then $\mathcal{V}$ is a base for the system of neighborhoods of e in (G,σ). Then $\{V \cap L : V \in \mathcal{V}\}$ is a base for the system of neighborhoods of e in (L,σ). Then as seen in Subsection 1.5,

$$N(L) = \bigcap\{(V \cap L)^{-1}(V \cap L) : V \in \mathcal{V}\}.$$

The proof is by contradiction. So assume that $H = L$. Then we have

$$L = KN(L) = N(L)K = \Big(\bigcap\{(V \cap L)^{-1}(V \cap L) : V \in \mathcal{V}\}\Big)K.$$

Hence for each $V \in \mathcal{V}$, $(V \cap L)^{-1}(V \cap L)K = L$ or equivalently, $(V \cap L)K$ is dense in (L,σ), *i.e.* $(U^{-1}U \cap L)K$ is open and dense in (L,σ) for each $U \in \mathcal{U}$. It follows from Lemma 3.1 that $(U^{-1}U \cap L)K$ is open and dense in (L,τ) for each $U \in \mathcal{U}$.

Since $K \neq L$, fix a point $a \in L \setminus K$. Since K is σ-closed, $(G/K,\tau)$ is Hausdorff by Lemma 1.3(c). Let $p : G \to G/K$ be the quotient map. Then since $a \notin K$, $p(a) \neq p(e)$. Hence there is a continuous function $r : (G/K,\tau) \to [0,1]$ such that $r(p(e)) = 0$ and $r \equiv 1$ on a τ-neighborhood of $p(a)$. Let $f = rp : G \to [0,1]$. Then f is continuous on (G,τ), $f(e) = 0$ and $f \equiv 1$ on a τ-neighborhood of $a \in L \setminus K$. Note that if $g \in G$ then $f(g) = f(gk)$ for each $k \in K$.

For the rest of the proof, the topology always refers to τ and we shall denote $\Lambda(G)$ by Λ. By induction on n, we construct a sequence $\{U_n : n \in \mathbb{N}\}$ in $\mathcal{U}$, a sequence $\{V_n : n \in \mathbb{N}\}$ of non-empty open subsets of G, each of which intersects L, and sequences $\{u_n : n \in \mathbb{N}\}$ and $\{x_n : n \in \mathbb{N}\}$ in G which satisfy the following conditions.

(i) $x_n \in U_{n-1}^{-1}U_{n-1}K \cap (V_{n-1} \cap \Lambda) = (U_{n-1} \cap \Lambda)^{-1}U_{n-1}K \cap (V_{n-1} \cap \Lambda)$, by Lemma 3.2.

(ii) $u_n \in U_{n-1} \cap \Lambda$.

(iii) $V_n \subset \overline{V_n} \subset V_{n-1} \subset f^{-1}(1)$ and $V_n \cap L \neq \emptyset$.

(iv) $u_n V_n \subset U_{n-1}K$.

(v) If H_n denotes the group generated by $\{u_1, x_1, u_2, x_2, \cdots, u_n, x_n\}$ and H_n is enumerated as $H_n = \{h_j^n : j \in \mathbb{N}\}$. Then $H_n \subset \Lambda$, $e \in U_n \subset \overline{U_n} \subset U_{n-1}$ and for each $t \in U_n$,

(3.5) $$|f(h_j^i t) - f(h_j^i)| \leq 1/n \quad \text{for} \quad 1 \leq i,j \leq n.$$

Construction. We let $U_0 = G$ and let V_0 be the interior of $f^{-1}(1)$ and u_0, x_0 are not defined. Assume that $n \in \mathbb{N}$ and that U_k, V_k are defined for $0 \leq k < n$ and x_k, u_k are defined for $0 < k < n$. By our assumptions there exists an $x \in (U_{n-1} \cap \Lambda)^{-1}U_{n-1}K \cap (V_{n-1} \cap L)$. So there is a $u_n \in U_{n-1} \cap \Lambda$ such that $u_n x \in U_{n-1}K$. Since $u_n \in \Lambda$, $x \in V_{n-1}$ and $U_{n-1}K$ is open, there is an open neighborhood V_n of x such that $x \in V_n \subset \overline{V_n} \subset V_{n-1}$ and $u_n V_n \subset U_{n-1}K$. Then $V_n \cap L \neq \emptyset$ since $x \in V_n \cap L$. Thus (ii)-(iv) are satisfied. Let x_n be any element of $V_n \cap \Lambda$, then by (iv) and (ii), (i) is satisfied and $H_n \subset \Lambda$ is defined. Finally since the map $t \mapsto |f(gt) - f(g)|$ is continuous for $g \in \Lambda$, an open neighborhood U_n of e satisfying (v) can be chosen. This completes the construction.

We let

$$U_\infty = \bigcap\{\overline{U_n} : n \in \mathbb{N}\} \quad \text{and} \quad H = \bigcup\{H_n : n \in \mathbb{N}\}$$

and let u_∞, x_∞ be cluster points of the sequences $\{u_n : n \in \mathbb{N}\}$, $\{x_n : n \in \mathbb{N}\}$ respectively. Clearly $u_\infty \in U_\infty$ and $x_\infty \in V_0$ and $\overline{H}$ is a subgroup of G, by Lemma 3.3. By the construction, (3.5) is satisfied for each $t \in \overline{U_n K}$. Hence $f(ht) = f(h)$ for each $h \in H$ and each $t \in \bigcap\{\overline{U_n K} : n \in \mathbb{N}\}$. Therefore, if we let

$$S = \{s \in \overline{H} : f(hs) = f(h) \text{ for each } h \in H\} = \{s \in \overline{H} : f(hs) = f(h) \text{ for each } h \in \overline{H}\},$$

then $\bigcap\{\overline{U_n K} : n \in \mathbb{N}\} \cap \overline{H} \subset S$ and S is a subgroup of G. By (ii), $u_\infty \in U_\infty \cap \overline{H} \subset S$ and by (iv) $u_n x_\infty \in \overline{U_{n-1}K} \cap \overline{H}$ for each $n \in \mathbb{N}$. Hence $u_\infty x_\infty \in \bigcap_{n\in\mathbb{N}} \overline{U_{n-1}K} \cap \overline{H} \subset S$. Therefore, $x_\infty = u_\infty^{-1}(u_\infty x_\infty) \in S^{-1}S \subset S$. Now, $f(s) = 0$ for all $s \in S$ since $f(es) = f(e) = 0$ for all $s \in S$. Therefore, $f(x_\infty) = 0$. However, since $x_\infty \in V_0 \subset f^{-1}(1)$, $f(x_\infty) = 1$. This contradiction completes the proof.

3.2 Haar measure

3.5 Corollary
Let (G, τ) be a CHART group and let σ be its σ-topology. Then there exists a transfinite sequence $\{L_\xi : 0 \le \xi \le \eta\}$ of subgroups of G such that

(a) *$L_0 = G$, $L_\eta = \{e\}$ and for each $\xi \le \eta$, L_ξ is a normal σ-closed subgroup of G;*

(b) *For $\xi < \eta$, $L_{\xi+1} \subset L_\xi$ and $(L_\xi / L_{\xi+1}, \tau)$ is a non-trivial compact Hausdorff topological group;*

(c) *The map*

$$m : (G/L_{\xi+1}, \tau) \times (L_\xi/L_{\xi+1}, \tau) \to (G/L_{\xi+1}, \tau)$$

defined by $m(xL_{\xi+1}, yL_{\xi+1}) = xyL_{\xi+1}$ for $x \in G$ and $y \in L_\xi$ is well-defined and continuous;

(d) *If ξ $(\le \eta)$ is a limit ordinal, then $L_\xi = \bigcap\{L_\zeta : 0 \le \zeta < \xi\}$.*

Proof. We define $\{L_\xi : 0 \le \xi \le \eta\}$ inductively in ξ. Let $L_0 = G$. Suppose $\nu \le \eta$ and assume inductively that $\{L_\xi : \xi < \nu\}$ have been defined so that (a), (d) are satisfied whenever $\xi < \nu$ and (b), (c) are true if $\xi + 1 < \nu$. If ν is a limit ordinal, then let $L_\nu = \bigcap\{L_\xi : \xi < \nu\}$. If ν is not a limit ordinal, then $\nu = \xi + 1$ and L_ξ is defined. If $L_\xi = \{e\}$, then let $\eta = \xi$ and stop the induction. If $L_\xi \ne \{e\}$, let $L_\nu = L_{\xi+1} = N(L_\xi)$. Then using Theorem 3.4 with $L = L_\xi$ and $K = \{e\}$, we see that $L_{\xi+1} \ne L_\xi$ and that $L_{\xi+1}$ is σ-closed in L_ξ and hence σ-closed in G by the inductive hypothesis. By Proposition 1.7(a) and the inductive hypothesis, $L_{\xi+1}$ is a normal subgroup of G and Proposition 1.7(b) shows the continuity property (c). The fact that $(L_\xi/L_{\xi+1}, \tau)$ is a compact Hausdorff space is shown in Subsection 1.5. Because $L_{\xi+1} \ne L_\xi$, this induction must come to a stop. This completes the proof.

Let (G, τ) be a compact Hausdorff right topological group.

A probability measure μ defined on the σ-algebra of Borel subsets of (G, τ) is called *right invariant* if $\mu(As) = \mu(A)$ for each $s \in G$ and Borel set A. Similarly, μ is called *left invariant* if $\mu(sA) = \mu(A)$ for each $s \in \Lambda(G)$ and each Borel set A. Following Milnes and Pym [6], we call a probability measure μ defined on the Borel subsets of G a *Haar measure* on (G, τ) if it is right invariant.

Milnes and Pym have shown that if a compact Hausdorff right topological group (G, τ) admits a transfinite sequence $\{L_\xi : 0 \le \xi \le \eta\}$ of subgroups of G satisfying conditions (a)–(d) of the Corollary 3.5, then (G, τ) has a unique Haar measure. The Haar measure is also left invariant. Hence we have the following corollary, which is also due to Milnes and Pym [7, Theorem 12].

3.6 Corollary
Each CHART group admits a unique Haar measure, which is also left invariant.

The next corollary was first proven in [4], in the case when X is metrizable, and then in the general case in [8, Corollary 4.1].

3.7 Corollary
Each distal flow (S, X) admits an S-invariant probability measure, i.e. a Borel probability measure μ on X such that $\mu(s \cdot A) = \mu(A)$ for each Borel subset A of X and each $s \in S$.

Proof. Let (G, τ) be the Ellis CHART group of the flow (S, X) and let λ the Haar measure on (G, τ). Let x_0 be any element of X and define the flow map $h : G \to X$ by, $h(g) = g(x_0)$, (see Section 2.1). Then consider the Borel probability measure μ on X defined by, $\mu(A) = \lambda(h^{-1}(A))$ for each Borel subset A of X. It is now routine to verify that μ is indeed S-invariant.

3.3 Inverse limit of compact flows

An *inverse system* $\{(S, X_\gamma), f_\alpha^\beta, \gamma \in \Gamma\}$ consists of a family $\{(S, X_\gamma) : \gamma \in \Gamma\}$ of compact flows indexed by a directed set Γ and a family $\{f_\alpha^\beta : \alpha, \beta \in \Gamma, \alpha \leq \beta\}$ of maps such that, for $\alpha, \beta \in \Gamma$ with $\alpha \leq \beta$, $f_\alpha^\beta : (S, X_\beta) \to (S, X_\alpha)$ is a flow map satisfying $f_\alpha^\beta f_\beta^\gamma = f_\alpha^\gamma$ whenever $\alpha, \beta, \gamma \in \Gamma$ and $\alpha \leq \beta \leq \gamma$. In this case, the *inverse limit* $\overleftarrow{\lim}\{(S, X_\gamma), f_\alpha^\beta, \gamma \in \Gamma\}$ is defined to be the compact flow (S, X), where

$$X = \Big\{x \in \prod_{\gamma \in \Gamma} X_\gamma : p_\alpha(x) = f_\alpha^\beta(p_\beta(x)) \text{ whenever } \alpha, \beta \in \Gamma \text{ and } \alpha \leq \beta\Big\}.$$

Here for each $\gamma \in \Gamma$, $p_\gamma : X \to X_\gamma$ is the γ^{th} projection and the action of S on X is given by

$$(s \cdot x)(\gamma) = s \cdot (x(\gamma)) \text{ for each } s \in S, x \in X \text{ and } \gamma \in \Gamma.$$

Now, let (G, τ) be a CHART group and let S be a subsemigroup of $\Lambda(G)$. Suppose that $\{K_\gamma : \gamma \in \Gamma\}$ is a family of σ-closed subgroups K_γ of G indexed by a directed set Γ in such a way that $K_\beta \subset K_\alpha$ whenever $\alpha, \beta \in \Gamma$ and $\alpha \leq \beta$. For each $\gamma \in \Gamma$, let $X_\gamma = (G/K_\gamma, \tau)$. Suppose the action of S on G is given by a semigroup homomorphism $\eta : S \to \Lambda(G)$ (see Subsection 1.2). Then we define the action of S on X_γ by $s \cdot gK_\gamma = \eta(s)gK_\gamma$ for each $s \in S$ and $\gamma \in \Gamma$. Then (S, X_γ) is a compact flow and we can make the family $\{(S, X_\gamma) : \gamma \in \Gamma\}$ into an inverse system of compact flows by defining the flow map $f_\alpha^\beta : X_\beta \to X_\alpha$ to be $gK_\beta \mapsto gK_\alpha$ for $\alpha, \beta \in \Gamma$ with $\alpha \leq \beta$. Then $\{(S, X_\gamma), f_\alpha^\beta, \gamma \in \Gamma\}$ is an inverse system of compact flows. We identify its inverse limit in the following lemma.

3.8 Lemma
Let the notation and symbols be as above. Let $K = \bigcap\{K_\gamma : \gamma \in \Gamma\}$. Then $(S, G/K)$ is isomorphic to the inverse limit $\overleftarrow{\lim}\{S, X_\gamma), f_\alpha^\beta, \gamma \in \Gamma\}$ of the system $\{(S, X_\gamma), f_\alpha^\beta, \gamma \in \Gamma\}$ of compact flows.

Proof. Let (S, X) denote the inverse limit of $\{(S, X_\gamma), f_\alpha^\beta, \gamma \in \Gamma\}$. Define $h : G/K \to \prod\{X_\gamma : \gamma \in \Gamma\}$ by $p_\gamma(h(gK)) = gK_\gamma$ for each $g \in G$ and $\gamma \in \Gamma$. Then it is obvious that $h(gK) \in X$ for each $g \in G$ and the map $h : G/K \to X$ is continuous. We must show that $h(G/K) = X$ and h is one-to-one. So suppose

$x \in X$. Then for each $\gamma \in \Gamma$, there exists $g_\gamma \in G$ such that $p_\gamma(x) = g_\gamma K_\gamma$. By the definition of the inverse limit, $p_\alpha(x) = f_\alpha^\beta(p_\beta(x))$ whenever $\alpha, \beta \in \Gamma$ and $\alpha \leq \beta$. This means that $g_\alpha K_\alpha = g_\beta K_\alpha$ or, equivalently, $g_\beta \in g_\alpha K_\alpha$ whenever $\alpha \leq \beta$. Let g_* be a cluster point of the net $\{g_\gamma : \gamma \in \Gamma\}$. Then $g_* \in g_\alpha K_\alpha$ for each $\alpha \in \Gamma$. It follows that for each $\gamma \in \Gamma$, $p_\gamma(h(g_* K)) = g_* K_\gamma = p_\gamma K_\gamma = p_\gamma(x)$ since $g_* \in p_\gamma K_\gamma$. Hence $h(g_* K) = x$ or $x \in h(G/K)$. Since $x \in X$ is arbitrary, $h(G/K) = X$. Now suppose $h(gK) = h(g_0 K)$ for some $g, g_0 \in G$. Then for each $\gamma \in \Gamma$, $gK_\gamma = g_0 K_\gamma$ and so $g^{-1} g_0 \in \bigcap\{K_\gamma : \gamma \in \Gamma\} = K$. Hence $gK = g_0 K$, whence h is one-to-one. It is easy to check that h is a flow map. This completes the proof.

3.4 The Furstenberg structure theorem

The next lemma is a generalization of Corollary 3.5.

3.9 Lemma
Let (G, τ) be a CHART group and let σ be its σ-topology. Suppose that K and L are σ-closed subgroups of G such that $K \subset L$ and $K \neq L$. Then there exists a transfinite sequence $\{H_\xi : 0 \leq \xi \leq \eta\}$ of σ-closed subgroups of G satisfying:

(a) $H_0 = L$, $H_\eta = K$ *and, for each* $\xi \leq \eta$, $K \subset H_\xi \subset L$*;*

(b) For each $\xi < \eta$, $H_{\xi+1} = N(H_\xi)K$ *and* $H_{\xi+1} \neq H_\xi$*;*

(c) If ξ $(\leq \eta)$ *is a limit ordinal, then* $H_\xi = \bigcap\{H_\zeta : 0 \leq \zeta < \xi\}$.

Proof. We define $\{H_\xi : 0 \leq \xi \leq \eta\}$ inductively in ξ starting with $H_0 = L$. Suppose that $\nu \leq \eta$ and assume inductively that $\{H_\xi : \xi < \nu\}$ have been defined so that (a), (c) are satisfied and (b) holds in case $\xi + 1 < \nu$. If ν is a limit ordinal then define $H_\nu = \bigcap\{H_\xi : \xi < \nu\}$. If ν has a predecessor, say ξ, then $\nu = \xi + 1$ and H_ξ has been defined. If $H_\xi = K$, then we let $\xi = \eta$ and the proof is finished. Otherwise, $H_\xi \neq K$ and H_ξ is σ-closed by the inductive hypothesis. Hence applying Theorem 3.4 to H_ξ and K (in the place of L and K), we conclude that $H_{\xi+1} := N(H_\xi)K$ is a σ-closed (in H_ξ hence in G) proper subgroup of H_ξ. Since $H_{\xi+1} \neq H_\xi$ as long as $H_\xi \neq K$, this induction must come to an end.

The next theorem, due to Ellis [3], is a generalization of the main result of [4].

3.10 Theorem (Furstenberg, Ellis)
Let (S, X) be a minimal distal flow and let (S, Y) be a factor (necessarily minimal distal) by a quotient flow map $\pi : (S, X) \to (S, Y)$. Then there exists an inverse system $\{(S, X_\xi), \pi_\alpha^\beta, 0 \leq \xi \leq \eta\}$ of minimal distal flows (indexed by ordinal numbers) having the following properties.

(i) $(S, X_0) = (S, Y)$ *and* $(S, X_\eta) = (S, X)$.

(ii) For each $\xi \leq \eta$, (S, X_ξ) *is a factor of* (S, X) *and an extension of* (S, Y).

(iii) For each $\xi < \eta$, *the flow map* $\pi_\xi^{\xi+1} : (S, X_{\xi+1}) \to (S, X_\xi)$ *is equicontinuous*

(iv) If ξ $(\leq \eta)$ *is a limit ordinal, then*

$$(S, X_\xi) = \overleftarrow{\lim}\{(S, X_\zeta), \pi_\alpha^\beta, 0 \leq \zeta < \xi\}.$$

Proof. Let (G,τ) be the Ellis group of the flow (S,X). Then by Proposition 2.1(ii), there exist σ-closed subgroups K and L of G such that $K \subset L$ and the flow map $\pi : (S,X) \to (S,Y)$ is represented by the map $\rho : (S, G/K) \to (S, G/L)$. Therefore we set $X = G/K,\ Y = G/L$ and use π in place of ρ. Recall that $\pi(=\rho)$ is the map $gK \mapsto gL$ for each $g \in G$. In order to avoid the trivial case, we assume that $K \neq L$.

Then by Lemma 3.9, we have a transfinite sequence $\{H_\xi : 0 \le \xi \le \eta\}$ of σ-closed subgroups of G satisfying (a)–(c) there. For each $\xi \in [0,\eta]$, let $(S,X_\xi) = (S,G/H_\xi)$ where $(S,G/H_\xi)$ is defined as in Lemma 1.4 and it is a minimal distal flow by that lemma. Property (a) implies (i) and (ii). For $0 \le \alpha \le \beta \le \eta$, define $\pi_\alpha^\beta : G/H_\beta(= X_\beta) \to G/H_\alpha(= X_\alpha)$ by $\pi_\alpha^\beta(gH_\beta) = gH_\alpha$. Then $\{(S,X_\xi), \pi_\alpha^\beta, 0 \le \xi \le \eta\}$ is an inverse system of minimal distal flows. Suppose that $\xi < \eta$. Then by (b), $H_{\xi+1} \neq H_\xi$ and $\pi_\xi^{\xi+1} : G/H_{\xi+1} \to G/H_\xi$ is equicontinuous by Theorem 2.2 since $N(H_\xi) \subset H_{\xi+1}$ by the definition of $H_{\xi+1}$. This proves (iii). Property (iv) follows from Lemma 3.9(c) and Lemma 3.8. This completes the proof.

References

[1] Robert Ellis, "Locally compact transformation groups", *Duke Math. J.* **24** (1957), 119–125.

[2] ______, "Distal transformation groups", *Pacific J. Math.* **8** (1958), 401–405.

[3] ______, "The Furstenberg structure theorem", *Pacific. J. Math.* **76** (1978), 345–349.

[4] H. Furstenberg, "The structure of distal flows", *Amer. J. Math.* **83** (1963), 477–515.

[5] Ryszard Engelking, *General Topology*, Monografie Matematyczne **60**, PWN-Polish Scientific Publishers, Warszawa, 1977.

[6] Paul Milnes and John Pym, "Haar measure for compact right topological groups", *Proc. Amer. Math. Soc.* **114** (1992), 387–393.

[7] ______, "Homomorphisms of minimal and distal flows", *J. Nigerian Math. Soc.* **11** (1992), 63–80. (http://www.math.uwo.ca/∼ milnes/cv/HAAR2.pdf)

[8] I. Namioka, " Right topological groups, distal flows and a fixed-point theorem", *Math. System Theory* **6** (1972), 193–209.

[9] ______, "Kakutani-type fixed point theorems, a survey", *J. Fixed Point Theory Appl.* **9** (2011), 1–23.

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