Some very rough notes on SEPARATE and JOINT CONTINUITY

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Contents

Chapter 1

Continuity and Games

1.1 Introduction to game theory

These unpublished notes are very rough and NOT proof-read or corrected. They were originally intended to be part of the monograph "Separate and Joint Continuity" by Jiling Cao and myself, published in Chapman & Hall/CRC Monographs and Research Notes in Mathematics, 2024, but were ultimately determined to be surplus to requirements.

Although a combinatorial game was described back at the beginning of the 17^{th} century, the notion of a *positional game* i.e., a two player game where the players alternate turns/moves in order to achieve a predefined winning condition) with *perfect information* (i.e., the players have available to them the same information concerning their next move, at the time of making that move, as they would have at the end of the game) was not formally introduced until the monograph of von Neumann and Morgenstern in 1944, [54]. In that monograph the authors considered finite positional games and proved that each such game can be reduced to a matrix game, and moreover, if the finite positional game is one with perfect information, then the corresponding matrix game has a saddle point. For a proof of this result and much more see, [54]. For more information on games also see [6,39,40,50,52].

However, infinite positional games with perfect information were discovered a little earlier. In 1935, Stanislaw Mazur proposed a game related to the Baire category theorem, which is described in Problem No. 43 of the Scottish book; its solution given by Stefan Banach is dated August 4, 1935. This game, now known as the Banach-Mazur game, is the first infinite positional game with perfect information. For more historical information on this game see, [50].

In this chapter we shall, for the most part, restrict ourselves to games that are essentially descendants of the Banach-Mazur game.

The first game that we shall consider is the Choquet game.

This game involves two players which we will call α and β. The "field/court" that the game is played on is a fixed topological space (X, τ). The name of the game is the *Choquet game* and is denoted by, $Ch(X)$.

After naming the game we need to describe how to "play" the $Ch(X)$-game. The player labeled β starts the game every time (life is not always fair). For his/her first move the player β must select nonempty open subset B_1 of X. Next, α gets a turn. For α's first move he/she must select a nonempty open subset A_1 of B_1. This ends the first round of the game.

In the second round, β goes first again and selects a nonempty open subset $B_2 \subseteq A_1$. Player α then gets to respond by choosing a nonempty open subset A_2 of B_2. This ends the second round of the game.

In general, after α and β have played the first n-rounds of the $Ch(X)$-game, β will have selected nonempty open subsets $B_1, B_2, \ldots, B_n$ and α will have selected nonempty open subsets $A_1, A_2, \ldots, A_n$ such that

$$A_n \subseteq B_n \subseteq A_{n-1} \subseteq B_{n-1} \subseteq \cdots \subseteq A_2 \subseteq B_2 \subseteq A_1 \subseteq B_1.$$

At the start of the $(n+1)$-round of the game, β goes first (again!) and selects nonempty open subset B_{n+1} of A_n. As with the previous n-rounds, the player α gets to respond to this move by selecting a nonempty open subset A_{n+1} of B_{n+1}.

Continuing this procedure indefinitely (i.e., continuing on forever) the players α and β produce an infinite sequence $\big((A_k, B_k) : k \in \mathbb{N}\big)$ called a *play* of the $Ch(X)$-game.

A *partial play* $\big((A_k, B_k) : 1 \leq k \leq n\big)$ of the $Ch(X)$-game consists of the first n-moves of a play of the $Ch(X)$-game.

As with any game, we need to specify a rule to determine who wins (otherwise, it is a very boring game). We shall declare that α *wins* a play $\big((A_k, B_k) : k \in \mathbb{N}\big)$ of the $Ch(X)$-game if: $\bigcap_{k\in\mathbb{N}} A_k = \bigcap_{k\in\mathbb{N}} B_k \neq \varnothing$.

If α does not win a play of the $Ch(X)$-game then we declare that β *wins* that play of the $Ch(X)$-game. So every play is won by either α or β and no play is won by both players.

Continuing further into game theory we need to introduce the notion of a strategy.

By a *strategy* t for the player β we mean a 'rule' that specifies each move of the player β in every possible situation. More precisely, a strategy $t := (t_n : n \in \mathbb{N})$ for β is an inductively defined sequence of τ-valued functions. The domain of t_1 is the sequence of length zero, denoted by $\emptyset$. That is, $\mathrm{Dom}(t_1) = \{\emptyset\}$ and $t_1(\emptyset) \in (\tau \setminus \{\varnothing\})$. If $t_1, t_2, \ldots, t_k$ have been defined then the domain of t_{k+1} is:

$$\{(A_1, \ldots, A_k) \in \tau^k : (A_1, \ldots, A_{k-1}) \in \mathrm{Dom}(t_k) \text{ and } \varnothing \neq A_k \subseteq t_k(A_1, \ldots, A_{k-1})\}.$$

For each $(A_1, A_2, \ldots, A_k) \in \mathrm{Dom}(t_{k+1})$, $t_{k+1}(A_1, A_2, \ldots, A_k) := B_{k+1} \in \tau$ is defined so that $\varnothing \neq B_{k+1} \subseteq A_k$.

A *partial* t*-play* is a finite sequence $(A_1, A_2, \ldots, A_n)$ such that $(A_1, A_2, \ldots, A_n) \in \mathrm{Dom}(t_{n+1})$. A t*-play* is an infinite sequence $(A_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $(A_1, A_2, \ldots, A_n)$ is a partial t-play.

A strategy $t := (t_n : n \in \mathbb{N})$ for the player β is called a *winning strategy* if each play of the form: $\big((A_n, t_n(A_1, \ldots, A_{n-1})) : n \in \mathbb{N}\big)$ is won by β.

Similarly we can define a strategy for α. By a *strategy* s for the player α we mean a 'rule' that specifies each move of the player α in every possible situation. More precisely, a strategy $s := (s_n : n \in \mathbb{N})$ for α is an inductively defined sequence of τ-valued functions. The domain of s_1 is $\{(B) : B \in \tau\backslash\{\varnothing\}\}$ and for each $B_1 \in \mathrm{Dom}(s_1)$, $s_1(B_1) := A_1 \in \tau$ is defined so that $\varnothing \neq A_1 \subseteq B_1$.

If $s_1, s_2, \ldots, s_k$ have been defined then the domain of s_{k+1} is:

$$\{(B_1, \ldots, B_{k+1}) \in \tau^{k+1} : (B_1, \ldots, B_k) \in \mathrm{Dom}(s_k) \text{ and } \varnothing \neq B_{k+1} \subseteq s_k(B_1, \ldots, B_k)\}.$$

For each $(B_1, B_2, \ldots, B_{k+1}) \in \mathrm{Dom}(s_{k+1})$, $s_{k+1}(B_1, B_2, \ldots, B_{k+1}) := A_{k+1} \in \tau$ is defined so that $\varnothing \neq A_{k+1} \subseteq B_{k+1}$.

A *partial s-play* is a finite sequence $(B_1, B_2, \ldots, B_n)$ such that $(B_1, B_2, \ldots, B_n) \in \mathrm{Dom}(s_n)$. An *s-play* is an infinite sequence $(B_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $(B_1, B_2, \ldots, B_n)$ is a partial s-play.

A strategy $s := (s_n : n \in \mathbb{N})$ for the player α is called a *winning strategy* if each play of the form: $\big((s_n(B_1, \ldots, B_n), B_n) : n \in \mathbb{N}\big)$ is won by α.

Note that since it is not possible for any play of the $Ch(X)$-game to be won by both players, it is not possible for both players to possess a winning strategy in the $Ch(X)$-game. Hence, if for example, the player α has a winning strategy in the $Ch(X)$-game then it is not possible for the player β to also have a winning strategy in the $Ch(X)$-game.

A space (X, τ) is called *weakly α-favourable* if α has a winning strategy in the $Ch(X)$-game.

Exercise 1.1.1. *Show that the following classes of topological spaces are weakly α-favourable in the Choquet game.*

(i) *Regular feebly compact spaces. Recall that a topological (X, τ) is called feebly compact if for every decreasing sequence $(U_n : n \in \mathbb{N})$ of nonempty open subsets of X, $\bigcap_{n\in\mathbb{N}} \overline{U_n}^{\tau} \neq \varnothing$. It is known that all completely regular feebly compact spaces are pseudo-compact i.e., every real-valued function defined on it is bounded, [12, p. 211].*

(ii) *All Čech-complete spaces (which includes all complete metric spaces), see [2, 15].*

Another important example, at least from the perspective of the study of separate and joint continuity, is the following example of a weakly α-favourable topological space.

Example 1.1.2. *Let Γ be an uncountable set. For each countable subset C of Γ and $f \in \Gamma^\Gamma$ let*

$$N(f, C) := \{g \in \Gamma^\Gamma : g|_C = f|_C\}.$$

Recall from Exercise 1.6.22 that $\{N(f, C) : f \in \Gamma^\Gamma$ and C is a countable subset of $\Gamma\}$ is a base for a topology on Γ^Γ which we called the topology of coincidence on countable sets and denoted τ_{count}. Then $(\Gamma^\Gamma, \tau_{count})$ is a weakly α-favourable space.

Proof. We shall inductively define a winning strategy $s := (s_n : n \in \mathbb{N})$ for the player α in the $Ch(X)$-game.

Step 1. Suppose that B_1 is a nonempty open subset of $(\Gamma^\Gamma, \tau_{count})$, i.e., we may think of B_1 as the first move of the player β. Choose $f^1_{(B_1)} \in B_1$ and a countable subset $C^1_{(B_1)}$ of Γ such that $N(f^1_{(B_1)}, C^1_{(B_1)}) \subseteq B_1$. Then define $s_1(B_1) := N(f^1_{(B_1)}, C^1_{(B_1)})$.

Now, let $n \in \mathbb{N}$ and suppose that s_j, f^j and C^j have been defined for every partial s-play $(B_1, \ldots, B_j)$ of length j with $1 \leq j \leq n$ so that:

(i) $f^j_{(B_1,\ldots,B_j)} \in B_j$;

(ii) $C^j_{(B_1,\ldots,B_j)}$ is a countable subset of Γ such that $N(f^j_{(B_1,\ldots,B_j)}, C^j_{(B_1,\ldots,B_j)}) \subseteq B_j$ and

(iii) $s_j(B_1,\ldots,B_j) := N(f^j_{(B_1,\ldots,B_j)}, C^j_{(B_1,\ldots,B_j)})$.

Step $n+1$. Let $(B_1,\ldots,B_{n+1})$ be a partial s-play of length $n+1$. Then B_{n+1} is a nonempty open subset of $s_n(B_1,\ldots,B_n) \subseteq B_n$. Choose $f^{n+1}_{(B_1,\ldots B_{n+1})} \in B_{n+1}$ and a countable subset $C^{n+1}_{(B_1,\ldots,B_{n+1})}$ of Γ such that $N(f^{n+1}_{(B_1,\ldots,B_{n+1})}, C^{n+1}_{(B_1,\ldots,B_{n+1})}) \subseteq B_{n+1}$. Then define,

$$s_{n+1}(B_1,\ldots,B_{n+1}) := N(f^{n+1}_{(B_1,\ldots,B_{n+1})}, C^{n+1}_{(B_1,\ldots,B_{n+1})}).$$

This completes the definition of $s := (s_n : n \in \mathbb{N})$.

So it remains to show that s is a winning strategy for the player α. To this end, let $(B_n : n \in \mathbb{N})$ be an arbitrary s-play. Let $f : \Gamma \to \Gamma$ be defined by, $f(\gamma) := f^m_{(B_1,\ldots,B_m)}(\gamma)$ if $\gamma \in \bigcup_{n\in\mathbb{N}} C^n_{(B_1,\ldots,B_n)}$ and m is the smallest natural number such that $\gamma \in C^m_{(B_1,\ldots,B_m)}$. If $\gamma \notin \bigcup_{n\in\mathbb{N}} C^n_{(B_1,\ldots,B_n)}$, then let $f(\gamma) := \gamma_0$, where γ_0 is some fixed element of Γ. We claim that $f \in \bigcap_{n\in\mathbb{N}} N(f^n_{(B_1,\ldots,B_n)}, C^n_{(B_1,\ldots,B_n)}) = \bigcap_{n\in N} B_n$. To substantiate this claim let us fix an $n \in \mathbb{N}$. We will show that $f \in N(f^n_{(B_1,\ldots,B_n)}, C^n_{(B_1,\ldots,B_n)})$. To this end, fix $\gamma \in C^n_{(B_1,\ldots,B_n)}$.

Let $m := \min\{k \in \mathbb{N} : \gamma \in C^k_{(B_1,\ldots,B_k)}\}$. Then $1 \le m \le n$ and $\gamma \in C^m_{(B_1,\ldots,B_m)}$. Therefore, by the definition of the function f, $f(\gamma) = f^m_{(B_1,\ldots,B_m)}(\gamma)$. On the other hand, since $m \le n$

$$N(f^n_{(B_1,\ldots,B_n)}, C^n_{(B_1,\ldots,B_n)}) \subseteq N(f^m_{(B_1,\ldots,B_m)}, C^m_{(B_1,\ldots,B_m)}).$$

In particular, $f^n_{(B_1,\ldots,B_n)} \in N(f^m_{(B_1,\ldots,B_m)}, C^m_{(B_1,\ldots,B_m)})$ and so

$$f(\gamma) = f^m_{(B_1,\ldots,B_m)}(\gamma) = f^n_{(B_1,\ldots,B_n)}(\gamma) \quad \text{as } \gamma \in C^m_{(B_1,\ldots,B_m)}.$$

Since $\gamma \in C^n_{(B_1,\ldots,B_n)}$ was arbitrary, $f \in N(f^n_{(B_1,\ldots,B_n)}, C^n_{(B_1,\ldots,B_n)})$. Furthermore, since $n \in \mathbb{N}$ was arbitrary, $f \in \bigcap_{n\in\mathbb{N}} N(f^n_{(B_1,\ldots,B_n)}, C^n_{(B_1,\ldots,B_n)})$. This completes the proof. □

Exercise 1.1.3. *Let Γ be an uncountable set and let $G^* := \{0,1\}^\Gamma$. Show that (G^*, τ_{count}) is weakly α-favourable. Hint: Modify the proof of Example 1.1.2.*

Exercise 1.1.4. *Let Γ be an uncountable set and let*

$$G_\Gamma := \{f \in \Gamma^\Gamma : f \textit{ is a bijection and } \{\gamma \in \Gamma : f(\gamma) \neq \gamma\} \textit{ is at most countable}\}.$$

Show that (G_Γ, τ_{count}) is a weakly α-favourable space. Hint: Modify the proof of Example 1.1.2.

Given that potentially, there are topological spaces (X, τ) where,

(i) the player β does not have a winning strategy and

(ii) the player α also fails to have a winning strategy, it makes sense to consider the following question.

"What topological spaces (X, τ) are characterised by the fact that the player β does not have a winning strategy in the $Ch(X)$-game?"

The answer to this important question has many fathers, but before we get onto that, let us first introduce some further notation that will expedite the proof of the following theorem. Let (X, τ) be a topological space and let $t := (t_n \in \mathbb{N})$ be a strategy for the player β in the $Ch(X)$-game played on (X, τ). If p is a partial t-play of the $Ch(X)$-game, then we define the *length of* p to be n if $p := (A_1, \ldots A_n)$ for some n, or 0, if $p = \emptyset$. Furthermore, if $p := (A_1, \ldots, A_n)$ is a partial t-play and A_{n+1} is a nonempty open subset of $t_{n+1}(A_1, \ldots, A_n)$, then we write (p, A_{n+1}) for the partial t-play $(A_1, \ldots, A_n, A_{n+1})$ of length $n+1$. Finally, if $p := (A_1, \ldots A_n)$ is a partial t-play of length n, for some $n \in \mathbb{N}$ and $1 \leq m < n$, then then we write $p|_m$ for the partial t-play $(A_1, \ldots, A_m)$ of length m. If $m = 0$, then $p|_m = \emptyset$.

Theorem 1.1.5 ([8, 35, 41, 45, 50])**.** *A topological space (X, τ) is a Baire space, (i.e., the intersection of every countable family of dense open sets is dense), if, and only if, the player β does not have a winning strategy in the $Ch(X)$-game.*

Proof. We shall first show that if β does not possess a winning strategy in the $Ch(X)$-game then (X, τ) is a Baire space. To do this, we shall prove the contrapositive statement. So let us suppose that (X, τ) is not a Baire space. Then there exists a sequence $(O_n : n \in \mathbb{N})$ of dense open subsets of X such that $\bigcap_{n\in\mathbb{N}} O_n$ is not dense in X. Therefore, there exists a nonempty open subset W of X such that $W \cap \bigcap_{n\in\mathbb{N}} O_n = \varnothing$. We shall now inductively define a winning strategy $t := (t_n : n \in \mathbb{N})$ for the player β. Let $t_1(\emptyset) := W$. Now, if $n \in \mathbb{N}$ and $(A_j : 1 \leq j \leq n)$ is any partial t-play of length n then we define $t_{n+1}(A_1, \ldots, A_n) := A_n \cap O_n$.

This defines a valid strategy $t = (t_n : n \in \mathbb{N})$ for β. Furthermore, t is clearly a winning strategy for the player β as for any t-play $(A_n : n \in \mathbb{N})$, $\bigcap_{n\in\mathbb{N}} A_n \subseteq W \cap \bigcap_{n\in\mathbb{N}} O_n = \varnothing$. This completes this direction of the proof.

Next, suppose that (X, τ) is a Baire space. Let $t := (t_n : n \in \mathbb{N})$ be any strategy for the player β in the $Ch(X)$-game. We need to show that there exists a t-play $(A_n : n \in \mathbb{N})$ where α wins, i.e., $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$. For each $n \in \mathbb{N}$, let P_n denote the set of all partial t-plays of length n and for purely notional reasons, let us also set $\Lambda_0 := \{\emptyset\}$ - the reason for this will become clear shortly. We shall inductively define a sequence $(\Lambda_n : n \in \mathbb{N})$ of subsets such that the following properties are fulfilled. For each $n \in \mathbb{N}$:

(a_n) $\Lambda_n \subseteq P_n$ and $t_{n+1}(p) \cap t_{n+1}(p') = \varnothing$ for every distinct $p, p' \in \Lambda_n$;

(b_n) $\bigcup\{t_{n+1}(p) : p \in \Lambda_n\}$ is dense in $t_1(\emptyset)$;

(c_n) for each $p \in \Lambda_n$, $p|_j \in \Lambda_j$ for all $0 \leq j < n$.

Step 1. Let Λ_1 be a maximal subset of P_1 with the property that $t_2(p) \cap t_2(p') = \varnothing$ for every distinct $p, p' \in \Lambda_1$ and $p|_0 \in \Lambda_0$ for every $p \in \Lambda_1$. By Zorn's Lemma such a maximal subset exists.

We claim that $\bigcup\{t_2(p) : p \in \Lambda_1\}$ is dense in $t_1(\emptyset)$. Indeed, if $\bigcup\{t_2(p) : p \in \Lambda_1\}$ is not dense in $t_1(\emptyset)$, then there exists a nonempty open subset A of $t_1(\emptyset)$ such that $\bigcup\{t_2(p) : p \in \Lambda_1\} \cap A = \varnothing$. Note that (A) is a partial t-play of length 1. Let $\Lambda^* := \Lambda_1 \cup \{(A)\}$. Then Λ^* satisfies Property (a_1) and Property (c_1). However, this contradicts the maximality of Λ_1. Hence, $\bigcup\{t_2(p) : p \in \Lambda_1\}$ must be dense in $t_1(\emptyset)$.

Let $n \in \mathbb{N}$, and suppose the subsets Λ_k satisfying the Properties (a_k), (b_k) and (c_k) have been defined for each $1 \leq k \leq n$.

Step $n+1$. Let Λ_{n+1} be a maximal subset of P_{n+1} with the property that $t_{n+2}(p) \cap t_{n+2}(p') = \varnothing$ for every distinct $p, p' \in \Lambda_{n+1}$ and $p|_j \in \Lambda_j$ for every $p \in \Lambda_{n+1}$ and every $0 \leq j < n+1$. By Zorn's Lemma such a maximal subset exists.

We claim that $\bigcup\{t_{n+2}(p) : p \in \Lambda_{n+1}\}$ is dense in $t_1(\emptyset)$. Indeed, if $\bigcup\{t_{n+2}(p) : p \in \Lambda_{n+1}\}$ is not dense in $t_1(\emptyset)$, then there exists a nonempty open subset A of $t_1(\emptyset)$ such that $\bigcup\{t_{n+2}(p) : p \in \Lambda_{n+1}\} \cap A = \varnothing$. Since, $\bigcup\{t_{n+1}(p) : p \in \Lambda_n\}$ is dense in $t_1(\emptyset)$ there exists a $p \in \Lambda_n$ such that $A' := t_{n+1}(p) \cap A \neq \varnothing$. Then (p, A') is a partial t-pay of length $n+1$. Let $\Lambda^* := \Lambda_{n+1} \cup \{(p, A')\}$. Then Λ^* satisfies Property (a_{n+1}) and Property (c_{n+1}). However, this contradicts the maximality of Λ_{n+1}. Hence, $\bigcup\{t_{n+2}(p) : p \in \Lambda_{n+1}\}$ must be dense in $t_1(\emptyset)$.

This completes the induction. For each $n \in \mathbb{N}$, let $W_n := \bigcup\{t_{n+1}(p) : p \in \Lambda_n\}$. By Property (b_n) we have that each set W_n is a dense open subset of $t_1(\emptyset)$ and since (X, τ) is a Baire space, $\bigcap_{n\in\mathbb{N}} W_n \neq \varnothing$. Let $x \in \bigcap_{n\in\mathbb{N}} W_n$ and let $n \in \mathbb{N}$. By Property (a_n) there exists a unique $p_n \in \Lambda_n$ such that $x \in t_{n+1}(p_n)$.

We claim that if $n < m$ then $p_m|_n = p_n$. To see this, first note that $p_m|_n \in \Lambda_n$, by Property (c_m), and secondly, that $x \in t_{n+1}(p_m|_n) \cap t_{n+1}(p_n)$. Therefore, by Property (a_n), it must be the case that $p_m|_n = p_n$. Thus, p_m is a continuation of the partial t-play.

Let $p := (A_n^n : n \in \mathbb{N})$, where for each $n \in \mathbb{N}$, $p_n := (A_1^n, \ldots, A_n^n)$. Clearly, $x \in \bigcap_{n\in\mathbb{N}} A_n^n$ as

$$x \in t_{n+1}(p_n) = t_{n+1}(A_1^n, \ldots, A_n^n) \subseteq A_n^n \quad \text{for all } n \in \mathbb{N}.$$

So it remains to show that p is a t-play. To this end, let $n \in \mathbb{N}$. Then

$$A_{n+1}^{n+1} \subseteq t_{n+1}(A_1^{n+1}, \ldots, A_n^{n+1}). \qquad (*)$$

Now, by above, for each $1 \leq j \leq n$, $p_{n+1}|_j = p_j$. Therefore, for each $1 \leq j \leq n$, $A_j^{n+1} = A_j^j$. Substituting this into Equation $(*)$ we get that

$$A_{n+1}^{n+1} \subseteq t_{n+1}(A_1^1, \ldots, A_n^n).$$

This shows that p is a t-play where α wins. □

Since every weakly α-favourable topological space (X, τ) is β-*unfavourable* (i.e., the player β does not possess a winning strategy in the $Ch(X)$-game played on (X, τ)). It follows, from Theorem 1.1.5, that all weakly α-favourable spaces are Baire spaces [though, one can directly show this, without too much bother, without recourse to Theorem 1.1.5]. However, the validity of the converse statement is not clear, i.e., are all Baire spaces weakly α-favourable? Or, equivalently, in terms of games, are all β-unfavourable spaces weakly α-favourable?

The answer to this question is no.

One way to see this, is to first show that if (X, τ) and (Y, τ') are both weakly α-favourable then so is $(X \times Y, \tau \times \tau')$, [8]. Here, $\tau \times \tau'$ denotes the product topology on $X \times Y$.

Since weakly α-favourable spaces are Baire spaces, the product $X \times Y$ will be a Baire space. However, it is known that there exist Baire spaces (X, τ) and (Y, τ') such that $(X \times Y, \tau \times \tau')$ is not a Baire space, [14]. These spaces are known as *barely Baire spaces*.

Hence, it follows that at least one of these spaces is a Baire space in which the player α does not possess a winning strategy in the $Ch(X)$-game, or, in light of Theorem 1.1.5, a space in which neither player, α nor β, has a winning strategy. Such games, where neither player has a winning strategy, are called *undetermined games*.

More simply, one can show that any *Bernstein subset* of $\mathbb{R}$, with the relative topology, is a Baire space that is not weakly α-favourable (i.e., neither player possesses a winning strategy, or if you prefer, the Choquet game on a Bernstein set is an undetermined game).

Recall that a subset B of $\mathbb{R}$ is called a *Bernstein set* if neither B nor its complement contains a perfect compact subset, [42, p.23]. In [42] the construction of a Bernstein set is given.

So, in summary, there are topological spaces (X, τ) in which neither player, α nor β, possesses a winning strategy in the $Ch(X)$-game played on (X, τ).

Exercise 1.1.6. *Let (X, τ) be a weakly α-favourable topological space and let U be a nonempty open subset of (X, τ). Show that U, equipped with the relative topology inherited from (X, τ), is also weakly α-favourable.*

Before we continue further into game theory, let us pause for a moment, to address the anxiety that you may be feeling.

The phrasing of results in terms of "players","winners" and "strategies" etc. probably seems very foreign to you, and you are most-likely deciding whether it is worth all the effort to learn this exotic area of mathematics, when all you want, is to learn about "Separate and Joint Continuity".

Well, let us try to alleviate your fears. Firstly, all the phrasing in terms of "games", "winners" and "strategies" etc. are just window dressing. Lying underneath these terms are basic notions from mathematics. In particular, everything concerning the Choquet game can be simply rephrased in terms of trees and induction. So if one really wants to, one can remove all the game terminology and replace it with more traditionally sounding terminology.

However, below are a few passages that hopefully convince you to "stick" with the game theory terminology.

The use of Banach-Mazur type games can often simplify the presentation of certain inductive arguments. One can design a game that exactly suits/fits the particular inductive argument under consideration. That is, the game can be tailor made to fit the situation.

The proof then divides into two parts. In one part we use the tailor made game to expedite the proof of the inductive argument. Strategies offering an effect way of recording the inductive hypotheses. The other part of the proof is then to determine those space/situations where the game conditions are satisfied. This dividing the proof into two parts is an important feature of the game approach - watch out for this in the future.

Another feature of the game formalism is the possibility of considering spaces where neither player possesses a winning strategy (see Theorem 1.1.5). Initially perhaps, it is not at all clear, how one might use the assumption: "I do not possess a winning strategy." However, the way in which one usually exploits the hypothesis/condition that β does not possess a winning strategy is the following:

One uses a proof by contradiction. That is, assume that the conclusion of the statement (that one wants to prove) is false. Then use this additional information to construct a strategy t for the player β. The fact that t is not a winning strategy for the player β then yields the *existence* of a play $\big((A_n, B_n) : n \in \mathbb{N}\big)$ where α wins. This play $\big((A_n, B_n) : n \in \mathbb{N}\big)$ is then used to obtain the required contradiction.

Games are used in many places within analysis. Some of these are listed below. The study of the Namioka property; the study of weak Asplund spaces and Gâteaux differentiability spaces; in the theory of selections (of set-valued mappings); in optimization of continuous and lower semi-continuous functions; in active boundaries of set-valued mappings (involves a game defined on filter bases); the study of closed graph theorems; the study of fragmentability and σ-fragmentability; in Baire category arguments; in differentiability theory; in the study semitopological groups/topological groups; Plus many other places.

1.2 Namioka spaces defined by games

For this section of the Chapter we will require the ensuing generalisation the the Stone-Weierstrass Theorem (see Theorem 1.6.18).

Corollary 1.2.1. *Let (Y, τ') be a compact topological space and let L be a sub-lattice of $C(Y)$. If $f \in \overline{L}^{\tau_p(Y)} \subseteq \mathbb{R}^Y$ and $\text{dist}(f, C(Y)) < \varepsilon$ then there exists an $l \in L$ such that $\|f - l\|_\infty < 2\varepsilon$. Here, $\overline{L}^{\tau_p(Y)}$ denote the closure of L in $(\mathbb{R}^Y, \tau_p(Y))$.*

Proof. Choose $g \in C(Y)$ such that $r := \|f - g\|_\infty < \varepsilon$. Then for each pair of points $x, y \in Y$, there exists an $l_{(x,y)} \in L$ such that $|f(x) - l_{(x,y)}(x)| < (\varepsilon - r)$ and $|f(y) - l_{(x,y)}(y)| < (\varepsilon - r)$. Hence,

$$|g(x) - l_{(x,y)}(x)| \leq |g(x) - f(x)| + |f(x) - l_{(x,y)}(x)| < r + (\varepsilon - r) = \varepsilon$$

and

$$|g(y) - l_{(x,y)}(y)| \leq |g(y) - f(y)| + |f(y) - l_{(x,y)}(y)| < r + (\varepsilon - r) = \varepsilon.$$

Therefore, by Theorem 1.6.18, there exists an $l \in L$ such that $\|g - l\|_\infty < \varepsilon$. Thus,

$$\|f - l\|_\infty \leq \|f - g\|_\infty + \|g - l\|_\infty < \varepsilon + \varepsilon = 2\varepsilon.$$

This completes the proof. □

Before we can state our most general theorem on Namioka spaces we will need to first recall the definition of a much studied class of topological spaces. Our formulation of this definition depends upon the notion of an "usco" mapping; which is what we present next.

Let (X, τ) and (Y, τ') be topological spaces. We shall say that a set-valued mapping $\Phi : X \to 2^Y$ is an *usco* on X if:

(i) $\Phi(x)$ is a nonempty compact subset of (Y, τ'), for each $x \in X$ and

(ii) for each open subset W of (Y, τ'), $\{x \in X : \Phi(x) \subseteq W\}$ is an open subset of (X, τ).

Building on this definition, we will say that a topological space (X, τ) is *$\mathcal{K}$-countably determined* if (i) (X, τ) is completely regular and (ii) there exists a separable metric space (S, d) and an usco mapping $\Phi : S \to 2^X$ such that $X = \Phi(S)$. This is, X is the usco image a separable metric space.

It is easy to see that all separable metric spaces and all compact Hausdorff spaces are $\mathcal{K}$-countably determined. On the other hand, it is reasonably straightforward to show that all $\mathcal{K}$-countably determined topological spaces satisfy the Lindelöf property.

Proposition 1.2.2. *Let (Y, τ') be a compact Hausdorff space and let $f : X \to C_p(Y)$ be a continuous function acting from a Baire space (X, τ). Suppose that $0 < \varepsilon$ is given. If for each nonempty open subset U of X there exists a nonempty open subset V of U and a $\mathcal{K}$-countably determined subset A of $C_p(Y)$ such that $f(V) \subseteq A + \varepsilon B_{C(Y)}$ then*

$$O_\varepsilon := \bigcup \{W \in \tau : \| \cdot \|_\infty - \operatorname{diam}[f(W)] \leq 8\varepsilon\}$$

is dense in (X, τ).

Proof. Suppose, in order to obtain a contradiction, that O_ε is not dense in (X, τ). Then there exists a nonempty open subset U of X such that $U \cap O_\varepsilon = \varnothing$. Note that without loss of generality we may assume that $f(U)$ is norm bounded in $C(Y)$. Indeed, if for each $n \in \mathbb{N}$, $U_n := \{x \in U : f(x) \in nB_{C(Y)}\}$, then $\{U_n : n \in \mathbb{N}\}$ is a closed cover of U. Since U is of the second Baire category there must exist a $k_0 \in \mathbb{N}$ such that $\operatorname{int}(U_{k_0}) \cap U \neq \varnothing$. If $U' := \operatorname{int}(U_{k_0}) \cap U$ then $\varnothing \neq U' \subseteq U$, $f(U')$ is norm bounded and $U' \cap O_\varepsilon = \varnothing$.

By the hypothesis there exists a nonempty open subset V of U and a $\mathcal{K}$-countably determined subset A of $C_p(Y)$ such that $f(V) \subseteq A + \varepsilon B_{C(Y)}$. Furthermore, by the definition of a $\mathcal{K}$-countably determined set there exists a separable metric space (S, d) and a $\tau_p(Y)$-usco $\Phi : S \to 2^{C(Y)}$ such that $A = \Phi(S)$. Let $(W_n : n \in \mathbb{N})$ be a countable base for the topology on (S, d). For each $n \in \mathbb{N}$, let

$$C_n := \overline{\Phi(W_n)}^{\tau_p(Y)} + \varepsilon B$$

where, $B := \{h \in \mathbb{R}^Y : |h(y)| \leq 1 \text{ for all } y \in Y\}$ and the closure is taken in $(\mathbb{R}^Y, \tau_p(Y))$.

We shall inductively define a (necessarily non-winning) strategy $t := (t_n : n \in \mathbb{N})$ for the player β in the Choquet-game played on (X, τ).

Base Step. Set $A_0 := U$, choose $l(\emptyset) \in f(A_0)$ and let $L(\emptyset)$ denote the finite lattice (in $C(Y)$) generated by $l(\emptyset)$. (In this case $L(\emptyset) = \{l(\emptyset)\}$.) Since $U \cap O_\varepsilon = \varnothing$, it follows that $f(U) \not\subseteq L(\emptyset) + (4\varepsilon)B_{C(Y)}$. Therefore, there exists a point $x \in A_0$ such that $f(x) \notin [L(\emptyset)+(4\varepsilon)B_{C(Y)}]$. Now, because: (i) $L(\emptyset)+(4\varepsilon)B_{C(Y)}$ is a closed set; (ii) the $\tau_p(Y)$-topology is regular and (iii) the function f is $\tau_p(Y)$-continuous, there exists an open neighbourhood V of x, contained in A_0, such that

$$\overline{f(V)}^{\tau_p(Y)} \cap [L(\emptyset) + (4\varepsilon)B_{C(Y)}] = \varnothing.$$

Now, by possibly making V smaller, we may assume that either, $\overline{f(V)}^{\tau_p(Y)} \subseteq C_1$, or else, $\overline{f(V)}^{\tau_p(Y)} \cap C_1 = \varnothing$. We then define, $t_1(\emptyset) := V$.

Now, suppose that the points $l(A_1, \ldots, A_{j-1})$, the finite sub-lattices $L(A_1, \ldots, A_{j-1})$ of $C(Y)$ and the strategies t_j have been defined for each partial t-play $(A_1, \ldots, A_{j-1})$ of length $(j-1)$ with $1 \leq j \leq n$ so that:

(A_j) $\overline{f(t_j(A_1, \ldots, A_{j-1}))}^{\tau_p(Y)} \cap [L(A_1, \ldots, A_{j-1}) + (4\varepsilon)B] = \varnothing$, where $L(A_1, \ldots, A_{j-1})$ denotes the finite sub-lattice of $C(Y)$ generated by $\{l(\emptyset), l(A_1), \ldots, l(A_1, \ldots, A_{j-1})\}$.

(B_j) either, $\overline{f(t_j(A_1, \ldots, A_{j-1}))}^{\tau_p(Y)} \subseteq C_j$, or else, $\overline{f(t_j(A_1, \ldots, A_{j-1}))}^{\tau_p(Y)} \cap C_j = \varnothing$.

Step $n+1$. Let $(A_1, \ldots, A_n)$ be a partial t-play of length n. Then

$$\varnothing \neq A_n \subseteq t_n(A_1, \ldots, A_{n-1}) \subseteq A_{n-1} \subseteq U.$$

Choose, $l(A_1, \ldots, A_n) \in f(A_n)$ and let $L(A_1, \ldots, A_n)$ denote the finite sub-lattice of $C(Y)$ generated by $\{l(\emptyset), l(A_1), \ldots, l(A_1, \ldots, A_n)\}$. Note that $L(A_1, \ldots, A_{n-1}) \subseteq L(A_1, \ldots, A_n)$.

Since $A_n \cap O_\varepsilon = \varnothing$ we have, by Lemma 1.6.21, that $f(A_n) \not\subseteq [L(A_1, \ldots, A_n) + (4\varepsilon)B_{C(Y)}]$. Therefore, there exists a point $x \in A_n$ such that $f(x) \notin [L(A_1, \ldots, A_n) + (4\varepsilon)B_{C(Y)}]$. Now, because: (i) $L(A_1, \ldots, A_n) + (4\varepsilon)B_{C(Y)}$ is a closed set (it is a finite union of closed balls); (ii) the $\tau_p(Y)$-topology is regular and (iii) the function f is $\tau_p(Y)$-continuous, there exists an open neighbourhood V of x, contained in A_n, such that

$$\overline{f(V)}^{\tau_p(Y)} \cap [L(A_1, \ldots, A_n)) + (4\varepsilon)B_{C(Y)}] = \varnothing.$$

Now, by possibly making V smaller, we may assume that either, $\overline{f(V)}^{\tau_p(Y)} \subseteq C_{n+1}$, or else, $\overline{f(V)}^{\tau_p(Y)} \cap C_{n+1} = \varnothing$. We then define, $t_{n+1}(A_1, \ldots, A_n) := V$. This completes the definition of $t := (t_n : n \in \mathbb{N})$.

Since (X, τ) is a Baire space we have, via Theorem 1.1.5, that t is not a winning strategy for the player β. Hence there exists a t-play $(A_n : n \in \mathbb{N})$ where α wins, i.e., $\bigcap_{n=1}^\infty A_n \neq \varnothing$

Let $g \in f(\bigcap_{n=1}^\infty A_n) \subseteq f(U) \subseteq A + \varepsilon B_{C(Y)} = \Phi(S) + \varepsilon B_{C(Y)}$. Therefore, there exists a point $s \in S$ such that $g \in \Phi(s) + \varepsilon B_{C(Y)} \subseteq \Phi(s) + \varepsilon B$.

Let $(W_{n_k} : k \in \mathbb{N})$ be a local base for the d-topology at $s \in S$. Fix $k \in \mathbb{N}$. Since,

$$g \in \Phi(s) + \varepsilon B \subseteq \overline{\Phi(W_{n_k})}^{\tau_p(Y)} + \varepsilon B = C_{n_k}$$

it follows from (B_{n_k}) that $\overline{f(t_{n_k}(A_1, \ldots, A_{n_k-1}))}^{\tau_p(Y)} \subseteq C_{n_k}$.

Let $l_\infty \in \mathbb{R}^Y$ be a $\tau_p(Y)$-cluster-point of $(l(A_1, \ldots, A_{n_k-1}) : k \in \mathbb{N})$. Note that such a cluster-point exists since the sequence $(l(A_1, \ldots, A_{n_k-1}) : k \in \mathbb{N})$ is pointwise bounded (in fact, uniformly bounded) in $\mathbb{R}^Y$. Furthermore,

$$\begin{aligned} l_\infty &\in \bigcap_{k=1}^\infty \overline{f(t_{n_k}(A_1, \ldots, A_{n_k-1}))}^{\tau_p(Y)} \text{ as } l(A_1, \ldots, A_{n_k-1}) \in f(t_{n_k}(A_1, \ldots, A_{n_k-1}))\ \forall k \in \mathbb{N} \\ &\subseteq \bigcap_{k=1}^\infty C_{n_k} \\ &= \bigcap_{k=1}^\infty [\overline{\Phi(W_{n_k})}^{\tau_p(Y)} + \varepsilon B] \\ &= \Phi(s) + \varepsilon B \quad \text{ since, } \Phi \text{ is an usco.} \end{aligned}$$

Thus, $\text{dist}(l_\infty, C(Y)) \leq \varepsilon < 2\varepsilon$. Now, since $L(A_1, \ldots, A_{n-1}) \subseteq L(A_1, \ldots, A_n)$ for all $n \in \mathbb{N}$, $\bigcup_{n=1}^\infty L(A_1, \ldots, A_n)$ is a sub-lattice of $C(Y)$. Therefore, by Corollary 1.2.1, there exists an $l \in \bigcup_{n=1}^\infty L(A_1, \ldots, A_n)$ such that $\|l_\infty - l\|_\infty < 4\varepsilon$ since, $l_\infty \in \overline{\bigcup_{n=1}^\infty L(A_1, \ldots, A_n)}^{\tau_p(Y)}$.

On the other hand, since

$$l_\infty \in \bigcap_{k=1}^\infty \overline{f(t_{n_k}(A_1, \ldots, A_{n_k-1}))}^{\tau_p(Y)} = \bigcap_{k=1}^\infty \overline{f(t_k(A_1, \ldots, A_{k-1}))}^{\tau_p(Y)}$$

we have, by Property (A_k), that $l_\infty \notin L(A_1, \ldots, A_{k-1}) + (4\varepsilon)B$, for each $k \in \mathbb{N}$. However, this contradicts the fact that $l \in L(A_1, \ldots, A_{k-1})$ for some $k \in \mathbb{N}$. Thus, it must be the case that O_ε is dense in (X, τ). □

Theorem 1.2.3. *Let (Y, τ') be a compact Hausdorff space and let $f : X \to C_p(Y)$ be a continuous function acting from a Baire space (X, τ). If, for each $0 < \varepsilon$ and nonempty open subset U of X, there exists a nonempty open subset V of U and a $\mathcal{K}$-countably determined subset A of $C_p(Y)$ such that $f(V) \subseteq A + \varepsilon B_{C(Y)}$, then f is norm continuous at the points of a dense and G_δ subset of (X, τ).*

Proof. Fix $0 < \varepsilon$ and consider the set

$$O_\varepsilon := \bigcup\{W \in \tau : \|\cdot\|_\infty - \text{diam}[f(W)] \leq \varepsilon\}.$$

Clearly, O_ε is open, as it is a union of open sets and, by Proposition 1.2.2 it follows that O_ε is also dense in (X, τ). Therefore, $\bigcap_{n\in\mathbb{N}} O_{\frac{1}{n}}$ is a dense and G_δ subset of (X, τ). So to complete the proof it only remains to observe that f is norm continuous at each point of $\bigcap_{n\in\mathbb{N}} O_{\frac{1}{n}}$. □

Corollary 1.2.4. *Let (Y, τ') be a compact Hausdorff space and let $f : X \to C_p(Y)$ be a continuous function acting from a Baire space (X, τ). If $C_p(Y)$ is $\mathcal{K}$-countably determined then f is norm continuous at the points of a dense and G_δ subset of (X, τ).*

Proof. This follows directly from Theorem 1.2.3. □

A compact Hausdorff space (Y, τ') for which $C_p(Y)$ is $\mathcal{K}$-countably determined is called a *Gul'ko compact*. These spaces were first considered in [18, 48] and then later in [53]. Since these spaces were first introduced they have been extensively studied, particularly in regard to renorming theory. Indeed, if $\mathcal{G}$ denotes that class of all Gul'ko compacta then it is known that $\mathcal{G}$ is stable under taking: continuous images; closed subspaces; countable products and finite unions, [48]. Furthermore, it is known that all Gul'ko compacta are Corson compact, [18] and that all Gul'ko compacta are fragmentable by a complete metric, [43]. It is also know that if $(Y, \tau') \in \mathcal{G}$ then $(B_{C(Y)^*}, \text{weak}^*) \in \mathcal{G}$, see [46]. For more information on $\mathcal{G}$ see [13, Chapter 7].

In order to say something more about the implications $\mathcal{K}$-countably determined sets have concerning norm continuity of pointwise continuous mappings, we need to explore further some of their basic properties.

Exercise 1.2.5. *This exercise concerns the basic properties of usco mappings.*

(a) *Let (X_1, τ_1), (X_2, τ_2), (Y_1, τ_1') and (Y_2, τ_2') be topological spaces. Show that if $\Phi_1 : X_1 \to 2^{Y_1}$ and $\Phi_2 : X_2 \to 2^{Y_2}$ are usco mappings then the mapping $\Phi : X_1 \times X_2 \to 2^{Y_1 \times Y_2}$ defined by,*

$$\Phi(x_1, x_2) := \Phi_1(x_1) \times \Phi_2(x_2) \quad \text{for all } (x_1, x_2) \in X_1 \times X_2$$

is an usco on $X_1 \times X_2$.

(b) *Let (X_i, τ_i), $i \in \mathbb{N}$ and (Y_i, τ_i'), $i \in \mathbb{N}$ be topological spaces. Show that if for each $i \in \mathbb{N}$ the mapping $\Phi_i : X_i \to 2^{Y_i}$ is an usco mapping then the mapping $\Phi : \prod_{i\in\mathbb{N}} X_i \to 2^{\prod_{i\in\mathbb{N}} Y_i}$ defined by,*

$$\Phi(x_1, x_2, \ldots, x_n, \ldots) := \prod_{i\in\mathbb{N}} \Phi_i(x_i) \quad \text{for all } (x_1, x_2, \ldots, x_n, \ldots) \in \prod_{i\in\mathbb{N}} X_i$$

is an usco on $\prod_{i\in\mathbb{N}} X_i$.

(c) *Let $\Phi : X \to 2^Y$ be an usco mapping acting from a topological space (X, τ) into subsets of a topological space (Y, τ') and let $f : Y \to Z$ be a continuous mapping from Y into a topological space (Z, τ''). Then the mapping $(f \circ \Phi) : X \to 2^Z$ defined by,*

$$(f \circ \Phi)(x) := \{f(y) \in Z : y \in \Phi(x)\} \quad \text{all } x \in X$$

is an usco on X.

Exercise 1.2.6. *We can now use Exercise 1.2.5 to deduce some facts concerning $\mathcal{K}$-countably determined spaces.*

(a) *Show that if (X, τ) and (Y, τ') are $\mathcal{K}$ countably determined topological spaces then so is $(X \times Y, \tau \times \tau')$. Hint: use Exercise 1.2.5 part (a).*

(b) *Show that if (X_i, τ_i), $i \in \mathbb{N}$ are $\mathcal{K}$-countably determined topological spaces then so is $\prod_{i\in\mathbb{N}} X_i$, endowed with the product topology. Hint: use Exercise 1.2.5 part (b).*

(c) *Show that if (X, τ) is a $\mathcal{K}$-countably determined topological space and $f : (X, \tau) \to (Y, \tau')$ is a continuous function into a completely regular space (Y, τ') then $f(X)$, with the relative topology, is a $\mathcal{K}$-countably determined space. Hint: use Exercise 1.2.5 part (c).*

(d) *Let (X, τ) be a completely regular topological space. If $(X_i : i \in \mathbb{N})$ are $\mathcal{K}$-countably determined subspaces of X then so is $\bigcup_{i\in\mathbb{N}} X_i$. Hint: note that (i) $\mathbb{N} \times \prod_{i\in\mathbb{N}} X_i$, endowed with the product topology, is a $\mathcal{K}$-countably determined space; (ii) if $f : \mathbb{N} \times \prod_{i\in\mathbb{N}} X_i \to X$ defined by, $f(m, (x_i)_{i=1}^{\infty}) := x_m$ for all $(m, (x_i)_{i=1}^{\infty}) \in \mathbb{N} \times \prod_{i\in\mathbb{N}} X_i$ then f is a continuous function; (iii) $f(\mathbb{N} \times \prod_{i\in\mathbb{N}} X_i) = \bigcup_{i\in\mathbb{N}} X_i$.*

We can now establish a very useful fact concerning $\mathcal{K}$-countably determined subspaces of $C(Y)$-spaces, in the case when (Y, τ') is a compact Hausdorff space.

Proposition 1.2.7. *Let (K, τ') be a compact Hausdorff topological space and let A be a $\mathcal{K}$-countably determined subspace of $C_p(Y)$. Then $L(A)$ - the smallest sub-lattice in $C(Y)$, containing A, is also a $\mathcal{K}$-countable determined topological space.*

Proof. Let us start by recalling that the functions $M : C_p(Y) \times C_p(Y) \to C_p(Y)$ and $m : C_p(Y) \times C_p(Y) \to C_p(Y)$ defined by, $M(f, g) := f \vee g$ and $m(f, g) := f \wedge g$ for all $(f, g) \in C(Y) \times C(Y)$, are continuous, see Exercise 1.6.17.

Let L be an arbitrary sub-lattice of $C(Y)$ containing the set A. We will inductively define an increasing sequence $(A_n : n \in \mathbb{N})$ of $\mathcal{K}$-countably determined subspaces of $C_p(Y)$.

Base Step. Let $A_0 := A$.

Next, suppose that $A_0, A_1, \ldots, A_n$ have been defined so that:

(i) $A_{j+1} := M(A_j \times A_j)$ if $0 \le j < n$ and j is even;

(ii) $A_{j+1} := m(A_j \times A_j)$ if $0 \le j < n$ and j is odd;

(iii) each A_j, with $0 \le j \le n$, is $\mathcal{K}$-countably determined;

(iv) $A_j \subseteq A_{j+1} \subseteq L$ for all $0 \le j < n$.

Step $n + 1$. If n is even then define $A_{n+1} := M(A_n \times A_n)$. If n is odd then define $A_{n+1} := m(A_n \times A_n)$. By Exercise 1.2.6 parts (a) and (c) we have, in both cases, that A_{n+1} is $\mathcal{K}$-countably determined. Furthermore, since $M(f, f) = f$ and $m(f, f) = f$ for all $f \in C(Y)$ we have that $A_n \subseteq A_{n+1}$. Finally, $A_{n+1} \subseteq L$ since $A_n \subseteq L$, by assumption, and L is a sub-lattice.

This completes the induction. Let $A_\infty := \bigcup_{n\in\mathbb{N}} A_n$. It follows from Properties (i), (ii) and (iv) that A_∞ is a sub-lattice of L. Since L was an arbitrary sub-lattice containing the set A we must have that A_∞ is the smallest sub-lattice in $C(Y)$, containing A. To see that A_∞ is $\mathcal{K}$-countably determined we just appeal to Property (iii) and Exercise 1.2.6 part (d). □

Corollary 1.2.8. *Let (Y, τ') be a compact Hausdorff space and let $f : X \to C_p(Y)$ be a continuous function acting from a Baire space (X, τ). If (X, τ) has a dense $\mathcal{K}$-countably determined subset then f is norm continuous at the points of a dense and G_δ subset of (X, τ).*

Proof. Let K be a dense, $\mathcal{K}$-countably determined subspace of (X, τ). Then, by Exercise 1.2.6 part (c) $f(K)$ is a $\mathcal{K}$-countably determined subspace of $C_p(Y)$. Therefore, by Proposition 1.2.7, $L(f(K))$ - the smallest sub-lattice in $C(Y)$ that contains $f(K)$, is $\mathcal{K}$-countably determined. Furthermore,

$$\begin{aligned} f(X) = f(\overline{K}^{\tau}) &\subseteq \overline{f(K)}^{\tau_p(Y)} \quad \text{since } f \text{ is } \tau_p(Y)\text{-continuous} \\ &\subseteq \overline{L(f(K))}^{\tau_p(Y)} = \overline{L(f(K))}^{\|\cdot\|_\infty} \quad \text{by, Exercise 1.6.19.} \end{aligned}$$

Thus, for each $0 < \varepsilon$, $f(X) \subseteq L(f(K)) + \varepsilon B_{C(Y)}$. The result now directly follows from Theorem 1.2.3. □

By appealing to game theory we can improve upon Corollary 1.2.8.

Let $\mathcal{P}$ be a nonempty collection of subsets of a topological space (X, τ). We shall use this collection of sets to generalise the Choquet game played on (X, τ). As before the game involves two players which we will call α and β. The game is played on the topological space (X, τ) and is called the $\mathcal{G}_{\mathcal{P}}$-*game* played on (X, τ).

As with the Choquet game we need to describe how to "play" the $\mathcal{G}_{\mathcal{P}}$-game. The player labeled β starts the game. For his/her first move the player β must select nonempty open subset B_1 of X. Next, α selects a pair (A_1, K_1) consisting of a nonempty open subset A_1 of B_1 and a set $K_1 \in \mathcal{P}$.

In the second round, β goes first again and selects a nonempty open subset $B_2 \subseteq A_1$. Player α then gets to respond by choosing a pair (A_2, K_2) consisting of a nonempty open subset A_2 of B_2 and a subset $K_2 \in \mathcal{P}$.

In general, after α and β have played the first n-rounds of the $\mathcal{G}_{\mathcal{P}}$-game, β will have selected nonempty open subsets $B_1, B_2, \ldots, B_n$ and α will have selected pairs

$$(A_1, K_1), (A_2, K_2), \ldots, (A_n, K_n)$$

consisting of nonempty open sets A_j of B_j and subsets $K_j \in \mathcal{P}$ such that

$$A_n \subseteq B_n \subseteq A_{n-1} \subseteq B_{n-1} \subseteq \cdots \subseteq A_2 \subseteq B_2 \subseteq A_1 \subseteq B_1.$$

At the start of the $(n+1)$-round of the game, β goes first and selects nonempty open subset B_{n+1} of A_n. Then player α gets to respond to this move by selecting a pair (A_{n+1}, K_{n+1}) consisting of a nonempty open subset A_{n+1} of B_{n+1} and $K_{n+1} \in \mathcal{P}$.

Continuing this procedure indefinitely (i.e., continuing on forever) the players α and β produce an infinite sequence $\big(((A_k, K_k), B_k) : k \in \mathbb{N}\big)$ called a *play* of the $\mathcal{G}_{\mathcal{P}}$-game.

A *partial play* $\big(((A_k, K_k), B_k) : 1 \leq k \leq n\big)$ of the $\mathcal{G}_{\mathcal{P}}$-game consists of the first n-moves of a play of the $\mathcal{G}_{\mathcal{P}}$-game.

We shall declare that α *wins* a play $\big(((A_k, K_k), B_k) : k \in \mathbb{N}\big)$ of the $\mathcal{G}_{\mathcal{P}}$-game if:

$$\left(\bigcap_{n\in\mathbb{N}} A_n\right) \cap \overline{\bigcup_{n\in\mathbb{N}} K_n}^{\tau} \neq \varnothing.$$

If α does not win a play of the $\mathcal{G}_{\mathcal{P}}$-game then we declare that β *wins* that play of the $\mathcal{G}_{\mathcal{P}}$-game. As with the Choquet game we need to define the notion of a strategy.

By a *strategy* t for the player β we mean a 'rule' that specifies each move of the player β in every possible situation. More precisely, a strategy $t := (t_n : n \in \mathbb{N})$ for β is an inductively defined sequence of τ-valued functions. The domain of t_1 is the sequence of length zero, denoted by $\emptyset$. That is, $\text{Dom}(t_1) = \{\emptyset\}$ and $t_1(\emptyset) \in (\tau \setminus \{\varnothing\})$. If $t_1, t_2, \ldots, t_k$ have been defined then the domain of t_{k+1} is:

$$\{((A_1, K_1), \ldots, (A_k, K_k)) \in (\tau \times \mathcal{P})^k : ((A_1, K_1), \ldots, (A_{k-1}, K_{k-1})) \in \text{Dom}(t_k) \text{ and } \varnothing \neq A_k \subseteq t_k(A_1, \ldots, A_{k-1})\}.$$

For each $((A_1, K_1), \ldots, (A_k, K_k)) \in \text{Dom}(t_{k+1})$, $t_{k+1}((A_1, K_1), \ldots, (A_k, K_k)) := B_{k+1} \in \tau$ is defined so that $\varnothing \neq B_{k+1} \subseteq A_k$.

A *partial t-play* is a finite sequence $((A_1, K_1), \ldots, (A_n, K_n))$ such that

$$((A_1, K_1), \ldots, (A_n, K_n)) \in \text{Dom}(t_{n+1}).$$

A *t-play* is an infinite sequence $((A_n, K_n) : n \in \mathbb{N})$ such that $((A_1, K_1), \ldots, (A_n, K_n))$ is a partial t-play for each $n \in \mathbb{N}$, .

A strategy $t := (t_n : n \in \mathbb{N})$ for the player β is called a *winning strategy* if each play of the form: $\big(((A_n, K_n), t_n((A_1, K_1), \ldots, (A_{n-1}, K_{n-1}))) : n \in \mathbb{N}\big)$ is won by β.

Similarly we can define a strategy for α. By a *strategy* s for the player α we mean a 'rule' that specifies each move of the player α in every possible situation. More precisely, a strategy $s := (s_n : n \in \mathbb{N})$ for α is an inductively defined sequence of $\tau \times \mathcal{P}$-valued functions. The domain of s_1 is $\big\{(B) : B \in \tau \setminus \{\varnothing\}\big\}$ and for each $B_1 \in \text{Dom}(s_1)$, $s_1(B_1) := (A_1, K_1) \in \tau \times \mathcal{P}$ is defined so that $\varnothing \neq A_1 \subseteq B_1$.

If $s_1, s_2, \ldots, s_k$ have been defined then the domain of s_{k+1} is:

$$\{(B_1, \ldots, B_{k+1}) \in \tau^{k+1} : (B_1, \ldots, B_k) \in \text{Dom}(s_k) \text{ and } \varnothing \neq B_{k+1} \subseteq A_k \text{ where } (A_k, K_k) := s_k(B_1, \ldots, B_k)\}.$$

For each $(B_1, B_2, \ldots, B_{k+1}) \in \text{Dom}(s_{k+1})$, $s_{k+1}(B_1, B_2, \ldots, B_{k+1}) := (A_{k+1}, K_{k+1}) \in \tau \times \mathcal{P}$ is defined so that $\varnothing \neq A_{k+1} \subseteq B_{k+1}$.

A *partial s-play* is a finite sequence $(B_1, B_2, \ldots, B_n)$ such that $(B_1, B_2, \ldots, B_n) \in \text{Dom}(s_n)$. An *s-play* is an infinite sequence $(B_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $(B_1, B_2, \ldots, B_n)$ is a partial s-play.

A strategy $s := (s_n : n \in \mathbb{N})$ for the player α is called a *winning strategy* if each play of the form: $\big((s_n(B_1, \ldots, B_n), B_n) : n \in \mathbb{N}\big)$ is won by α.

The following result, while not surprising, serves to demonstrate the structure of game theoretic proofs.

Proposition 1.2.9. *Let $\mathcal{P}$ be a nonempty collection of subsets of a (X, τ) be a topological space. If (X, τ) is β unfavourable in the $\mathcal{G}_{\mathcal{P}}$-game played on (X, τ) (i.e., the player β does not possess a winning strategy in the $\mathcal{G}_{\mathcal{P}}$-game), then (X, τ) is a Baire space.*

Proof. We shall consider the converse statement. To that end, let us suppose that (X, τ) is not a Baire space. Then, by Theorem 1.1.5, the player β has a winning strategy $t := (t_n : n \in \mathbb{N})$ in the $Ch(X)$-game, i.e., $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$ for every t-play $(A_n : n \in \mathbb{N})$. We shall use the strategy t to inductively construct a winning strategy $t' := (t'_n : n \in \mathbb{N})$ for the player β in the $\mathcal{G}_{\mathcal{P}}$-game played on (X, τ). This will then show that (X, τ) is β favourable in the $\mathcal{G}_{\mathcal{P}}$-game (i.e., *not* β unfavourable).

Step 1. Let $t'_1(\emptyset) := t_1(\emptyset)$.

Now, let $n \in \mathbb{N}$ and suppose that t'_j have been defined for every partial t'-play

$$((A_1, K_1), \ldots, (A_{j-1}, K_{j-1}))$$

of length $(j-1)$ with $1 \leq j \leq n$ so that: $(A_1, \ldots, A_{j-1})$ is a partial t-play and

$$t'_j((A_1, K_1), \ldots, (A_{j-1}, K_{j-1})) := t_j(A_1, \ldots, A_{j-1}).$$

Step $n+1$. Let $((A_1, K_1), \ldots, (A_n, K_n))$ be a partial t'-play of length n. Then,

$$A_n \subseteq t'_n((A_1, K_1), \ldots, (A_{n-1}, K_{n-1})) = t_n(A_1, \ldots, A_{n-1}) \subseteq A_{n-1}.$$

Since, by assuption $(A_1, \ldots, A_{n-1})$ is a partial t-play and $A_n \subseteq t_n(A_1, \ldots, A_{n-1})$ we have that $(A_1, \ldots, A_n)$ is a partial t-play of length n. In particular, $t_{n+1}(A_1, \ldots, A_n)$ is defined. Let $t'_{n+1}((A_1, K_1), \ldots, (A_n, K_n)) := t_{n+1}(A_1, \ldots, A_n)$. This completes the definition of $t' := (t'_n : n \in \mathbb{N})$.

We now show that t' is indeed a winning strategy for the player β in the $\mathcal{G}_{\mathcal{P}}$-game. To this end, let $((A_n, K_n) : n \in \mathbb{N})$ be an arbitrary t'-play. By construction $(A_n : n \in \mathbb{N})$ is a t-play and so $\bigcap_{n \in \mathbb{N}} A_n = \varnothing$. Thus, in particular, $(\bigcap_{n \in \mathbb{N}} A_n) \cap \overline{\bigcup_{n \in \mathbb{N}} K_n}^{\tau} = \varnothing$. This shows that t' is a winning strategy for the player β in the $\mathcal{G}_{\mathcal{P}}$-game played on (X, τ). $\square$

Theorem 1.2.10 ([44])**.** *Let (Y, τ') be a compact Hausdorff space and let $f : X \to C_p(Y)$ be a continuous function acting from a topological space (X, τ). If (X, τ) is β-unfavourable in the $\mathcal{G}_{cd}$-game played on (X, τ), then f is norm continuous at the points of a dense and G_δ subset of (X, τ).*

Proof. We shall suppose, in order to obtain a contradiction, that

$$C(f) := \{x \in X : f \text{ is norm continuous at } x\}$$

is not residual in (X, τ). Then, by Theorem 1.2.3, there must exist an $0 < \varepsilon$ and a nonempty open subset U of X such that, for every nonempty open subset V of U and every $\mathcal{K}$-countably determined subset A of $C_p(Y)$, there exists a point $x \in V$, such that $f(x) \notin A + \varepsilon B_{C(Y)}$.

We will use this assumption to create a (necessarily non-winning) strategy $t := (t_n : n \in \mathbb{N})$ for the player β in the $\mathcal{G}_{cd}$-game played on (X, τ).

Base Step. Set $A_0 := U$, $K_0 := \{t\}$ for some $t \in A_0$ and let $L(\emptyset)$ denote the lattice generated by $f(K_0)$. (In this case $L(\emptyset) := \{f(t)\}$.) By assumption $f(A_0) \not\subseteq L(\emptyset) + \varepsilon B_{C(Y)}$. Therefore, there exists a point $x \in A_0$ such that $f(x) \notin L(\emptyset) + \varepsilon B_{C(Y)}$. Now, because $L(\emptyset) + \varepsilon B_{C(Y)}$ is $\tau_p(Y)$-closed there exists an open neighbourhood V of x, contained in A_0, such that $f(V) \cap [L(\emptyset) + \varepsilon B_{C(Y)}] = \varnothing$. Then we define $t_1(\emptyset) := V$.

Now, suppose that the strategies t_j have been defined for each partial t-play

$$((A_1, K_1), \ldots, (A_{j-1}, K_{j-1}))$$

of length $(j-1)$ with $1 \leq j \leq n$ so that:

(A_j) $t_j((A_1, K_1), \ldots, (A_{j-1}, K_{j-1}))$ is a nonempty open subset of A_{j-1};

(B_j) $f(t_j((A_1, K_1), \ldots, (A_{j-1}, K_{j-1}))) \cap [L((A_1, K_1), \ldots, (A_{j-1}, K_{j-1})) + (\varepsilon/2)B_{C(Y)}] = \varnothing$, where $L((A_1, K_1), \ldots, (A_{j-1}, K_{j-1}))$ denotes the lattice generated by,

$$f(K_0) \cup f(K_1) \cup \cdots \cup f(K_{j-1}).$$

Step $n+1$. Let $((A_1, K_1), \ldots, (A_n, K_n))$ be a partial t-play of length n. Then

$$\varnothing \neq A_n \subseteq t_n((A_1, K_1), \ldots, (A_{n-1}, K_{n-1})) \subseteq U$$

and K_n is a $\mathcal{K}$-countably determined subset of (X, τ). Let $L((A_1, K_1), \ldots, (A_n, K_n))$ denote the lattice generated by, $f(K_0) \cup f(K_1) \cdots \cup f(K_n)$. By Exercise 1.2.6 part (a), part (c) and Proposition 1.2.7 we see that $L((A_1, K_1), \ldots, (A_n, K_n))$ is $\mathcal{K}$-countably determined. Hence, by assumption, $f(A_n) \nsubseteq [L((A_1, K_1), \ldots, (A_n, K_n)) + \varepsilon B_{C(Y)}]$. Therefore, there exists a point $x \in A_n$ such that $f(x) \notin [L((A_1, K_1), \ldots, (A_n, K_n)) + \varepsilon B_{C(Y)}]$. Thus, by Lemma **??**, $f(x) \notin \overline{[L((A_1, K_1), \ldots, (A_n, K_n)) + (\varepsilon/2)B_{C(Y)}]}^{\tau_p(Y)}$. It now follows from the $\tau_p(Y)$-continuity of f that there exists an open neighbourhood V of x, contained in A_n such that

$$f(V) \cap \overline{[L((A_1, K_1), \ldots, (A_n, K_n)) + (\varepsilon/2)B_{C(Y)}]}^{\tau_p(Y)} = \varnothing.$$

We now define $t_{n+1}((A_1, K_1), \ldots, (A_n, K_n)) := V$. This then completes the definition of $t := (t_n : n \in \mathbb{N})$.

Since (X, τ) is β-unfavourable in the $\mathcal{G}_{cd}$-game, t is not a winning strategy. Hence there exists a t-play $((A_n, K_n) : n \in \mathbb{N})$ where α wins, i.e., $(\bigcap_{n\in\mathbb{N}} A_n) \cap \overline{\bigcup_{n\in\mathbb{N}} K_n}^{\tau} \neq \varnothing$. Let $t \in (\bigcap_{n\in\mathbb{N}} A_n) \cap \overline{\bigcup_{n\in\mathbb{N}} K_n}^{\tau}$. Then,

$$\begin{aligned} f(t) \in f\left(\overline{\bigcup_{n\in\mathbb{N}} K_n}^{\tau}\right) &\subseteq \overline{f(\bigcup_{n\in\mathbb{N}} K_n)}^{\tau_p(Y)} \quad \text{since, } f \text{ is } \tau_p(Y)\text{-continuous} \\ &= \overline{\bigcup_{n\in\mathbb{N}} f(K_n)}^{\tau_p(Y)} \\ &\subseteq \overline{\bigcup_{n\in\mathbb{N}} L((A_1, K_1), \ldots, (A_n, K_n))}^{\tau_p(Y)}. \end{aligned}$$

Since $L((A_1, K_1), \ldots, (A_n, K_n)) \subseteq L((A_1, K_1), \ldots, (A_{n+1}, K_{n+1}))$ for all $n \in \mathbb{N}$,

$$\bigcup_{n\in\mathbb{N}} L((A_1, K_1), \ldots, (A_n, K_n))$$

is a sub-lattice of $C(Y)$. Thus, by Exercise 1.6.19, $f(t) \in \overline{\bigcup_{n\in\mathbb{N}} L((A_1, K_1), \ldots, (A_n, K_n))}^{\|\cdot\|_\infty}$. In particular, there exists an $l \in L((A_1, K_1), \ldots, (A_m, K_m))$ for some $m \in \mathbb{N}$, such that $\|f(t) - l\|_\infty < \varepsilon/2$. However, this contradicts Property (B_{m+1}) since

$$t \in t_{m+1}((A_1, K_1), \ldots, (A_m, K_m)) \quad \text{and} \quad \|f(t) - l\|_\infty < \varepsilon/2.$$

Thus, it must be the case that f is norm continuous at the points of a dense and G_δ subset of (X, τ). □

When Theorem 1.2.10 came out it generalised several earlier results which we now present.

In the following Corollary the $\mathcal{G}_p$-game will denote the $\mathcal{G}_{\mathcal{P}}$-game when $\mathcal{P}$ is the set of all singleton subsets of (X, τ); the $\mathcal{G}_K$-game will denote the $\mathcal{G}_{\mathcal{P}}$-game when $\mathcal{P}$ is the set of all compact subsets of (X, τ) and the $\mathcal{G}_{ka}$-game will denote the $\mathcal{G}_{\mathcal{P}}$-game when $\mathcal{P}$ is the set of all $\mathcal{K}$-analytic subsets of (X, τ). Recall that s subset K of a topological space (X, τ) is $\mathcal{K}$-*analytic* if it is the usco image of $\mathbb{N}^{\mathbb{N}}$, endowed with the Baire metric, [48].

Corollary 1.2.11. *Let (Y, τ') be a compact Hausdorff space and let $f : X \to C_p(Y)$ be a continuous function acting from a Baire space (X, τ).*

(i) *If (X, τ) is β-unfavourable in the $\mathcal{G}_p$-game played on (X, τ),* [45] *or*

(ii) *(X, τ) is β-unfavourable in the $\mathcal{G}_K$-game played on (X, τ),* [49] *or*

(iii) *(X, τ) is β-unfavourable in the $\mathcal{G}_{ka}$-game played on (X, τ),* [9]

then f is norm continuous at the points of a dense and G_δ subset of (X, τ).

One of the short comings of Theorem 1.2.10 is that it is not particularly easy to directly show that a given topological space is β-unfavourable in the $\mathcal{G}_{cd}$-game played on (X, τ). One way to over come this problem is to consider Baire spaces where the player α has a strategy that is "almost" a winning strategy. In this way one can show, in a more natural way, that many Baire spaces are actually β-unfavourable in the $\mathcal{G}_{cd}$-game,

Let $\mathcal{P}$ be a nonempty collection of subsets of a topological space (X, τ). We shall say that a topological space (X, τ) is *conditionally α-favourable in the $\mathcal{G}_{\mathcal{P}}$-game* if the player α possesses a strategy $s := (s_n : n \in \mathbb{N})$ such that for every s-play $(B_n : n \in \mathbb{N})$ either, $\bigcap_{n\in\mathbb{N}} B_n = \varnothing$ or else, $(\bigcap_{n\in\mathbb{N}} B_n) \cap \overline{\bigcup_{n\in\mathbb{N}} K_n} \neq \varnothing$, where $(A_n, K_n) := s_n(B_1, \ldots, B_n)$ for all $n \in \mathbb{N}$.

Theorem 1.2.12. *Let $\mathcal{P}$ be a nonempty collection of subsets of a Baire space (X, τ). If (X, τ) is conditionally α-favourable in the $\mathcal{G}_{\mathcal{P}}$-game then, (X, τ) is β-unfavourable in the $\mathcal{G}_{\mathcal{P}}$-game played on (X, τ).*

Proof. We shall start by introducing some notation. Let $\pi : 2^X \times 2^X \to 2^X$ be defined by, $\pi(A, B) := A$ for all $(A, B) \in 2^X \times 2^X$. Let $t := (t_n : n \in \mathbb{N})$ be an arbitrary strategy for the player β in the $\mathcal{G}_{cd}$-game played on (X, τ). Our goal is to construct a t-play where player α wins.

Let $s := (s_n : n \in \mathbb{N})$ be a strategy for the player α in the $\mathcal{G}_{cd}$-game played on (X, τ) such that for every s-play $(B_n : n \in \mathbb{N})$ either, $\bigcap_{n\in\mathbb{N}} B_n = \varnothing$ or else $\overline{\bigcup_{n\in\mathbb{N}} K_n} \cap \bigcap_{n\in\mathbb{N}} B_n \neq \varnothing$, where $(A_n, K_n) := s_n(B_1, \ldots, B_n)$ for all $n \in \mathbb{N}$.

We shall now consider the $Ch(X)$-game. Specifically, we shall inductively define a strategy $t' := (t'_n : n \in \mathbb{N})$ for the player β in the $Ch(X)$-game.

Step 1. Define $t'_1(\emptyset) := \pi(s_1(t_1(\emptyset)))$.

Now, let $n \in \mathbb{N}$ and suppose that t'_j have been defined for every partial t'-play $(A_1, \ldots, A_{j-1})$ of length $(j - 1)$ with $1 \leq j \leq n$ so that:

(i) $(A_1, \ldots, A_{j-1})$ is a partial t-play and

(ii) $(t_1(\emptyset), \ldots, t_j(A_1, \ldots, A_{j-1}))$ is a partial s-play and

(iii) $t'_j(A_1, \ldots, A_{j-1}) := \pi(s_j(t_1(\emptyset), \ldots, t_j(A_1, \ldots, A_{j-1})))$.

Step $n+1$. Let $(A_1, \ldots, A_n)$ be a partial t'-play of length n. Then

$$A_n \subseteq t'_n(A_1, \ldots, A_{n-1}) = \pi(s_n(t_1(\emptyset), \ldots, t_n(A_1, \ldots, A_{n-1}))) \subseteq t_n(A_1, \ldots, A_{n-1}).$$

Since, by assumption, $(A_1, \ldots, A_{n-1})$ is a partial t-play and $A_n \subseteq t_n(A_1, \ldots, A_{n-1})$ we have that $(A_1, \ldots, A_n)$ be a partial t-play. Therefore, $t_{n+1}(A_1, \ldots, A_n)$ is defined. Moreover,

$$t_{n+1}(A_1, \ldots, A_n) \subseteq A_n \subseteq \pi(s_n(t_1(\emptyset), \ldots, t_n(A_1, \ldots, A_{n-1}))). \quad (*)$$

Since, by assumption, $(t_1(\emptyset), \ldots, t_n(A_1, \ldots, A_{n-1}))$ is a partial s-play and by $(*)$ we have that $(t_1(\emptyset), \ldots, t_{n+1}(A_1, \ldots, A_n))$ is a partial s-play. We then define

$$t'_{n+1}(A_1, \ldots, A_n) := \pi(s_{n+1}(t_1(\emptyset), \ldots, t_{n+1}(A_1, \ldots, A_n))).$$

This completes the definition of $t' := (t'_n : n \in \mathbb{N})$. Since (X, τ) is a Baire space we have, via Theorem 1.1.5, that there is a t'-play $(A_n : n \in \mathbb{N})$ of the $Ch(X)$-game where α wins, i.e., $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$. Since, by the construction of the strategy t', we have that $(t_n(A_1, \ldots, A_{n-1}) : n \in \mathbb{N})$ is an s-play, we must have that $\overline{\bigcup_{n\in\mathbb{N}} K_n} \cap \bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$, where $(B_n, K_n) := s_n(A_1, \ldots, A_n)$ for all $n \in \mathbb{N}$, as $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$. Thus, again by the construction of the strategy t', $(A_n : n \in \mathbb{N})$ is a t-play. Hence, $(A_n : n \in \mathbb{N})$ is a t-play where α wins, in the $\mathcal{G}_{\mathcal{P}}$-game played on (X, τ). This completes the proof. □

The previous theorem is useful because there is a large class of spaces that are easily shown to be conditionally α-favourable in the $\mathcal{G}_{cd}$-game.

Exercise 1.2.13. *Let (X, τ) be a topological space. Show that (X, τ) is conditionally α-favourable in the $\mathcal{G}_{cd}$-game played on (X, τ) if:*

(i) *(X, τ) is pointwise countably complete (which includes all metric spaces, p-spaces and all Čech-complete spaces);*

(ii) *(X, τ) is fragmented by a metric whose topology on X is at least as strong as τ;*

(iii) *(X, τ) contains, as a dense subset, a $\mathcal{K}$-countably determined subspace (which includes all separable spaces).*

For the class of completely regular conditionally α-favourable spaces we have the following characterisation of Namioka spaces.

Theorem 1.2.14. *Let (X, τ) be a completely regular topological space that is conditionally α-favourable in the $\mathcal{G}_{cd}$-game played on (X, τ). Then (X, τ) is a Namioka space if, and only if, (X, τ) is a Baire space.*

Proof. Suppose that (X, τ) is a completely regular Namioka space. Then, by Theorem **??**, (X, τ) is a Baire space. Conversely, if (X, τ) is conditionally α-favourable in the $\mathcal{G}_{cd}$-game and a Baire space then, by Theorem 1.2.12, (X, τ) is β-unfavourable in the $\mathcal{G}_{\mathcal{P}}$-game played on (X, τ). Therefore, by Theorem 1.2.10, (X, τ) is a Namioka space. □

To further increase the significance of Theorem 1.2.10 we will show that, in addition to the earlier examples of conditionally α-favourable, in the $\mathcal{G}_{cd}$-game, spaces there is another large class of topological spaces that are also conditionally α-favourable, in the $\mathcal{G}_{cd}$-game.

Let (X, τ) be a Hausdorff topological space and let $Y \subseteq X$. A family α of subsets of X is said to *separate* Y *from* $X \setminus Y$ if, for every pair of points $y \in Y$ and $x \in X \setminus Y$, there is some $H \in \alpha$ which contains one of the points (no matter which one) and does not contain the other.

Since the space (X, τ) is Hausdorff the family of all open sets in X certainly separates Y from $X \setminus Y$, whatever $Y \subseteq X$. Among all families α of open sets which separate Y from $X \setminus Y$ there is one with least cardinality. It is this least cardinal number which we denote by $s_X(Y)$ and call the *separation index of* Y *in* X or, simply, *separation of* Y *in* X, [32, p. 205]. For example, if the set Y is open (or closed) in X, then $s_X(Y) = 1$ because the set Y (or its complement) separates Y from $X \setminus Y$. More generally, since every family α that separates Y from $X \setminus Y$ also separates $X \setminus Y$ from $Y = X \setminus (X \setminus Y)$, we have $s_X(Y) = s_X(X \setminus Y)$.

If a family α of open sets separates Y from $X \setminus Y$, then so does also the family of closed sets $\{X \setminus H : H \in \alpha\}$. Therefore, in the definition of $s_X(Y)$ one could take families of closed sets or even "mixed" families of sets which are either closed or open.

There is a close relation between $s_X(Y)$ and $s_{\overline{Y}}(Y)$, the separation index of Y in $\overline{Y}$. Clearly, $s_{\overline{Y}}(Y) \leq s_X(Y)$. On the other hand, the open set $X \setminus \overline{Y}$ can be added to any family α of closed subsets of $\overline{Y}$ that separates Y from $\overline{Y} \setminus Y$. The new family $\alpha' = \alpha \cup \{X \setminus \overline{Y}\}$ separates Y from $X \setminus Y$. Hence $s_X(Y) \leq s_{\overline{Y}}(Y) + 1$. In particular, if $s_{\overline{Y}}(Y)$ is infinite, then $s_X(Y) = s_{\overline{Y}}(Y)$. Therefore, the important case is when $X = \overline{Y}$. As will become clear later in this section, every completely regular space (Y, τ') has one and the same separation index in every compactification of Y. This is why we will write $s(Y)$ instead of $s_X(Y)$ and will use the term *separation of* Y instead of *separation of* Y *in* X.

Given an infinite cardinal number s we denote by $\Theta(X, s)$ the family of all subsets $Y \subseteq X$ for which $s_X(Y) \leq s$. $\Theta(X, s)$ contains all open subsets and all closed subsets of X. In the special case when $s = \aleph_0$, we say that the members of $\Theta(X, s)$ have *countable separation in* X. It is easy to see that if s is an infinite cardinal number then the family $\Theta(X, s)$ is closed under taking countable unions, countable intersections and the Souslin operation. In particular, for every infinite cardinal s the family $\Theta(X, s)$ contains all Borel subsets of X and all other subsets that can be obtained from the Borel sets by means of the Souslin operation. Therefore, all such sets have countable separation index in X. Other examples of subsets with countable separation are given by the following exercise.

Exercise 1.2.15. *Let* (Y, τ) *be the continuous image of some separable metric space* (Z, d). *Then* $s_X(Y)$ *is at most countable for every regular Hausdorff space* (X, τ') *containing* Y. *Hint: Let* $(U_i : i \in \mathbb{N})$ *be a countable topological base for* (Z, d) *and let* $h : Z \to Y$ *be a continuous mapping onto* Y. *Verify that the family* $(\overline{h(U_i)} : i \in \mathbb{N})$ *separates* Y *from* $X \setminus Y$.

We now present one of the fundamental properties of the separation index.

Theorem 1.2.16 ([32]). *Let (X_1, τ) and (Y_1, τ') be compact Hausdorff spaces and let $h : X_1 \to Y_1$ a continuous mapping from X_1 onto Y_1. Suppose X and Y are respectively, subsets of X_1 and Y_1, such that $Y = f(X)$ and $X = h^{-1}(Y)$. Then either, $s_{X_1}(X)$ and $s_{Y_1}(Y)$ are both finite, or else, both infinite. In the latter case, $s_{X_1}(X) = s_{Y_1}(Y)$.*

Proof. Let α be a family of open subsets of Y_1 which separates Y from $Y_1 \setminus Y$. Consider the family $\alpha' = \{H' \in 2^{X_1} : H' = h^{-1}(H)$ for some $H \in \alpha\}$ which consists of open subsets of X_1. It is clear that α' separates X from $X_1 \setminus X$. This shows that $s_{X_1}(X) \leq s_{Y_1}(Y)$.

We will now prove that the inverse inequality also holds. Suppose α is a family of closed subsets of X_1 that separates X from $X_1 \setminus X$. Without loss of generality we may assume that the family α is closed under taking finite intersections. If this is not the case, then we can add to α all such intersections. If α was an infinite family, the new family will have the same infinite cardinality. If α was finite, the new family will be finite again. We will prove now that $h(\alpha) := \{h(F) : F \in \alpha\}$ (a family consisting of closed subsets of Y_1) separates Y from $Y_1 \setminus Y$. Indeed, let $a \in Y$, $b \in Y_1 \setminus Y$ and put $A := h^{-1}(a)$, $B := h^{-1}(b)$. Let us assume, for the purpose of obtaining a contradiction, that a and b cannot be separated by sets of the type $h(F)$, where $F \in \alpha$. Then the families $\alpha_A := \{F \in \alpha : F \cap A \neq \varnothing\}$ and $\alpha_B := \{F \in \alpha : F \cap B \neq \varnothing\}$ coincide. Indeed, if there is some $F \in \alpha_A \setminus \alpha_B$, then $h(F)$ would contain a but not b and we would have the separation. Similarly if there is some $F \in \alpha_B \setminus \alpha_A$, then $h(F)$ would contain b but not a and we would have the separation in this case too.

Put $\alpha^* := \alpha_A = \alpha_B$. Note that α^* is a nonempty family. Indeed, take some $u \in A \subseteq X$ and $v \in B \subseteq X_1 \setminus X$. Then there exists an $F \in \alpha$ which contains one of the points u, v but not the other. If $u \in F$ then $F \in \alpha_A = \alpha^*$ and if $v \in F$ then $F \in \alpha_B = \alpha^*$. So in either case, $F \in \alpha^*$.

Denote by Δ the set of all subfamilies $\delta \subseteq \alpha^*$ for which $\bigcap\{F : F \in \delta\} \cap A \neq \varnothing$. For every finite $\delta' \subseteq \delta \in \Delta$ the set $\bigcap\{F : F \in \delta'\}$ belongs to α (as α is closed under finite intersections) and intersects A. Therefore, $\bigcap\{F : F \in \delta'\}$ is a member of α_A. Since $\alpha_A = \alpha_B$ it follows that $\bigcap\{F : F \in \delta'\} \cap B \neq \varnothing$. Since B is compact, this implies that $\bigcap\{F : F \in \delta\} \cap B \neq \varnothing$.

Next, note that $\Delta \neq \varnothing$. To see this, simply take $F \in \alpha^* \neq \varnothing$ and note that $\{F\} \in \Delta$.

Order Δ by set inclusion. Using the compactness of the set A once again we see that the union of any increasing chain of elements of Δ is again in Δ. Thus, by Zorn's lemma, there exists a family $\delta_{max} \subseteq \alpha^*$, which is a maximal element of $(\Delta, \subseteq)$.

There exist points $u_0 \in \bigcap\{F : F \in \delta_{max}\} \cap A \subseteq X$ and $v_0 \in \bigcap\{F : F \in \delta_{max}\} \cap B \subseteq X_1 \setminus X$.

Since α separates the points u_0 and v_0, there is some $F_0 \in \alpha$ which contains just one of these points. We consider the two cases.

If $u_0 \in F_0$ then $F_0 \in \alpha_A = \alpha^*$ and the family $\{F_0\} \cup \delta_{max}$ belongs to Δ. This implies, because of the maximality of δ_{max}, that $F_0 \in \delta_{max}$. Therefore, $v_0 \in F_0$ which is the desired contradiction.

If $v_0 \in F_0$ then $F_0 \in \alpha_B = \alpha^*$ and the family $\{F_0\} \cup \delta_{max}$ belongs to Δ. To see this last assertion consider the following. For every finite $\delta^{'} \subseteq \delta_{max}$ the set $\bigcap\{F : F \in \delta^{'}\} \cap F_0$ belongs to α (as α is closed under finite intersections) and intersects B, indeed $v_0 \in \bigcap\{F : F \in \delta^{'}\} \cap F_0 \cap B$. Therefore, $\bigcap\{F : F \in \delta^{'}\} \cap F_0 \in \alpha_B$. Since $\alpha_B = \alpha_A$, $\bigcap\{F : F \in \delta^{'}\} \cap F_0 \cap A \neq \varnothing$. As A is compact, this implies that $\bigcap\{F : F \in \delta_{max}\} \cap F_0 \cap A \neq \varnothing$. Thus, $\{F_0\} \cup \delta_{max} \in \Delta$.

However, because of the maximality of δ_{max}, we must have that $F_0 \in \delta_{max}$. Therefore, $u_0 \in F_0$, which is the desired contradiction.

Since, in both cases, we obtained our desired contradiction, we have proven the claim. □

Theorem 1.2.17 ([32]). *Let (X, τ) be a completely regular space. If X has finite a separation index in some compact Hausdorff space, then it has finite separation index in any compact Hausdorff space. If X has infinite separation index in some compact Hausdoff space, then it has the same separation index in any compact Hausdorff space. In particular, if $(\beta X, \tau_\beta)$ is the Stone-Čech compactification of (X, τ), then $s_{bX}(X) = s_{\beta X}(X)$ for every Hausdorff compactification (bX, τ_b) of (X, τ).*

Proof. We appeal to Theorem 1.2.16 with $X = Y$, $X_1 := \beta X$ and $Y_1 := bY$, for some compactification of Y and h is the unique continuous extension of the identity mapping $i : X \to Y$ to a mapping from βX onto bY, □

The above theorem allows us to use the terms "separation index of X" or simply, "separation of X" without mentioning the larger compact Hausdorff space where the separation index of X is measured. Correspondingly, we will denote this index simply by $s(X)$.

The next assertion shows further that the separation index is preserved under perfect mappings, i.e., continuous surjective mappings that send closed sets to closed sets and have the property that every point in their codomain has a compact preimage.

Corollary 1.2.18. *Let $h : X \to Y$ be a perfect mapping from a completely regular space (X, τ) onto a completely regular space (Y, τ'). Then $s(X)$ and $s(Y)$ are simultaneously finite or infinite. In the latter case (i.e., both infinite) we have that $s(X) = s(Y)$.*

Proof. Let $h : X \to Y$ be a perfect mapping and let $h^* : \beta X \to \beta Y$ be the unique continuous extension of h to βX. Then: (i) h^* is continuous; (ii) h^* is surjective and (iii) $h^*(X) = h(X) = Y$. So, to apply Theorem 1.2.16, we need only show that $(h^*)^{-1}(Y) = X$. In fact, because $X = h^{-1}(Y) \subseteq (h^*)^{-1}(Y)$ we need only show that $(h^*)^{-1}(Y) \subseteq X$. Therefore, in order to obtain a contradiction, let us assume that $(h^*)^{-1}(Y) \nsubseteq X$. Then there exists a point $x \in (h^*)^{-1}(Y) \setminus X$.

Let $y := h^*(x) \in Y$. Now, $h^{-1}(y)$ is a nonempty compact subset of X. Since $x \notin h^{-1}(y) \subseteq X$ there exist disjoint open subsets W and V of βX such that $h^{-1}(y) \subseteq W$ and $x \in V$. Since h is a perfect mapping there exists an open neighbourhood U of y in βY such that $h^{-1}(U \cap Y) \subseteq W \cap X \subseteq W$. Now, since h^* is continuous and $h^*(x) = y \in U$ there exists an open neighbourhood V' of x, contained in V, such that $h^*(V') \subseteq U$. Let $x' \in V' \cap X \neq \varnothing$. Then $h(x') = h^*(x') \in U$ and so $x' \in h^{-1}(U \cap Y) \subseteq W$. However, this contradicts the fact that $x' \in V' \subseteq V$ and $V \cap W = \varnothing$. Hence $(h^*)^{-1}(Y) \subseteq X$. The result now follows from Theorem 1.2.16. □

Theorem 1.2.19. *Let (X,τ) be a completely regular topological space. If (X,τ) has a countable separation index then (X,τ) is conditionally α-favourable in the $\mathcal{G}_p$-game (and hence in the $\mathcal{G}_{cd}$-game) played on (X,τ).*

Proof. Let $(O_n : n \in \mathbb{N})$ be a countable family of open subsets of $(\beta X, \tau_\beta)$ that separate X from $\beta X \setminus X$ and let $\pi_1 : 2^X \times 2^X \to 2^X$ and $\pi_2 : 2^X \times 2^X \to 2^X$ be defined by,

$$\pi_1(A,B) := A \quad \text{and} \quad \pi_2(A,B) := B \quad \text{for all } (A,B) \in 2^X \times 2^X.$$

We shall inductively define a strategy $s := (s_n : n \in \mathbb{N})$ for the player α in the $\mathcal{G}_p$-game played on (X,τ).

Step 1. Suppose that B_1 is a nonempty open subset of (X,τ), i.e., we may think of B_1 as the first move of the player β. If $B_1 \cap O_1 = \varnothing$ then choose $x \in B_1$ and an open neighbourhood U of x such that $x \in U \subseteq \overline{U}^\tau \subseteq B_1$. Then define $s_1(B_1) := (U, \{x\})$. On the other hand, if $B_1 \cap O_1 = \varnothing$ then choose $x \in B_1 \cap O_1$ and an open neighbourhood U of x such that $x \in U \subseteq \overline{U}^{\tau_\beta} \subseteq O_1$ and $\overline{U}^\tau \subseteq B_1$. Then define $s_1(B_1) := (U, \{x\})$. Note that in both cases: $\overline{\pi_1(s_1(B_1))}^\tau \subseteq B_1$ and either $\overline{\pi_1(s_1(B_1))}^{\tau_\beta} \subseteq O_1$ or $\overline{\pi_1(s_1(B_1))}^{\tau_\beta} \cap O_1 = \varnothing$.

Now, let $n \in \mathbb{N}$ and suppose that s_j have been defined for every partial s-play $(B_1, \ldots, B_j)$ of length j with $1 \leq j \leq n$ so that:

(i) $\overline{\pi_1(s_j(B_1, \ldots, B_j))}^\tau \subseteq B_j$;

(ii) either, $\overline{\pi_1(s_j(B_1, \ldots, B_j))}^{\tau_\beta} \subseteq O_j$ or $\overline{\pi_1(s_j(B_1, \ldots, B_j))}^{\tau_\beta} \cap O_j = \varnothing$.

Step $n+1$. Let $(B_1, \ldots, B_{n+1})$ be a partial s-play of length $n+1$. If $B_{n+1} \cap O_{n+1} = \varnothing$ then choose $x \in B_{n+1}$ and an open neighbourhood U of x such that $x \in U \subseteq \overline{U}^\tau \subseteq B_{n+1}$. Then define $s_{n+1}(B_1, \ldots, B_{n+1}) := (U, \{x\})$. On the other hand, if $B_{n+1} \cap O_{n+1} = \varnothing$ then choose $x \in B_{n+1} \cap O_{n+1}$ and an open neighbourhood U of x such that $x \in U \subseteq \overline{U}^{\tau_\beta} \subseteq O_{n+1}$ and $\overline{U}^\tau \subseteq B_{n+1}$. Then define $s_{n+1}(B_1, \ldots B_{n+1}) := (U, \{x\})$. Note that in both cases: $\overline{\pi_1(s_{n+1}(B_1, \ldots, B_{n+1}))}^\tau \subseteq B_{n+1}$ and either $\overline{\pi_1(s_{n+1}(B_1, \ldots, B_{n+1}))}^{\tau_\beta} \subseteq O_{n+1}$ or $\overline{\pi_1(s_{n+1}(B_1, \ldots, B_{n+1}))}^{\tau_\beta} \cap O_{n+1} = \varnothing$.

This completes the definition of $s := (s_n : n \in \mathbb{N})$. So it remains to show that s is a conditionally winning strategy for the player α in the $\mathcal{G}_p$-game played on (X,τ). To this end, let $(B_n : n \in \mathbb{N})$ be an arbitrary s-play with $\bigcap_{n\in\mathbb{N}} B_n \neq \varnothing$.

Let $x_\infty \in \beta X$ be a cluster-point of the sequence $(\pi_2(s_n(B_1, \ldots, B_n)) : n \in \mathbb{N})$. Since $\pi_2(s_k(B_1, \ldots, B_k)) \in B_k \subseteq B_n$ for all $n \leq k$ we have that $x_\infty \in \bigcap_{n\in\mathbb{N}} \overline{B_n}^{\tau_\beta}$. Further, since the points of $C := \bigcap_{n\in\mathbb{N}} \overline{B_n}^{\tau_\beta}$ are not distinguished by the sets $(O_n : n \in \mathbb{N})$ either $C \subseteq X$ or $C \subseteq \beta X \setminus X$. However, as $\varnothing \neq \bigcap_{n\in\mathbb{N}} B_n \subseteq C \cap X$, $C \subseteq X$. In particular, $x_\infty \in X$. Therefore, x_∞ is a cluster-point of the sequence $(\pi_2(s_n(B_1, \ldots, B_n)) : n \in \mathbb{N})$ in (X,τ). Thus,

$$x_\infty \in \bigcap_{n\in\mathbb{N}} \overline{\pi_2(s_n(B_1, \ldots, B_n))}^\tau \subseteq \bigcap_{n\in\mathbb{N}} \overline{B_n}^\tau = \bigcap_{n\in\mathbb{N}} B_n.$$

This shows that s is a conditionally winning strategy for the player α in the $\mathcal{G}_p$-game played on (X,τ). □

Also, later (in the comments section) mention the generalisation V. V. Mykhaylyuk in BAMS 2006. Mention results of Pol and Chaber (and later Moors and Lin). Also result of A. Bouziad and A. Bareche.

1.3 co-Namioka spaces defined by games

We shall introduce a new game that is somewhat different to the Choquet and $\mathcal{G}_{\mathcal{P}}$-game.

As with the earlier games this game involves two players which we will call α and β.

Given a topological space (X, τ) and a dense subset D of (X, τ) the $\mathcal{G}_D(\Delta)$-*game* is played on $(X \times X, \tau \times \tau)$.

As with the Choquet game and the $\mathcal{G}_{\mathcal{P}}$-game we need to describe how to "play" the $\mathcal{G}_D(\Delta)$-game. The player labeled β starts the game. For his/her first move the player β must select an open subset B_1 of $X \times X$ that contains $\Delta := \{(x, y) \in X \times X : x = y\}$. Next, α selects a point $(x_1, y_1) \in D \times D$ such that $(x_1, y_1) \in B_1$.

In the second round, β goes first again and selects an open subset B_2 of $X \times X$ that contains Δ. Player α then gets to respond by choosing a point $(x_2, y_2) \in D \times D$ such that $(x_2, y_2) \in B_2$.

In general, after α and β have played the first n-rounds of the $\mathcal{G}_D(\Delta)$-game, β will have selected open subsets $B_1, B_2, \ldots, B_n$ of $X \times X$, containing Δ and α will have selected points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \quad \text{in } D \times D$$

such that $(x_j, y_j) \in B_j$ for all $1 \leq j \leq n$.

At the start of the $(n+1)$-round of the game, β goes first and selects an open subset B_{n+1} of $X \times X$ that contains Δ. Then player α gets to respond to this move by selecting a point $(x_{n+1}, y_{n+1}) \in D \times D$ such that $(x_{n+1}, y_{n+1}) \in B_{n+1}$.

Continuing this procedure indefinitely (i.e., continuing on forever) the players α and β produce an infinite sequence $\big((B_k, (x_k, y_k)) : k \in \mathbb{N}\big)$ called a *play* of the $\mathcal{G}_D(\Delta)$-game.

A *partial play* $\big((B_k, (x_k, y_k)) : 1 \leq k \leq n\big)$ of the $\mathcal{G}_D(\Delta)$-game consists of the first n-moves of a play of the $\mathcal{G}_D(\Delta)$-game.

We shall declare that β *wins* a play $\big((B_k, (x_k, y_k)) : k \in \mathbb{N}\big)$ of the $\mathcal{G}_D(\Delta)$-game if for every open neighbourhood U of Δ, $\{k \in \mathbb{N} : (x_k, y_k) \in U\}$ is infinite. Note that if (X, τ) is a compact space, then the sequence $((x_k, y_k) : k \in \mathbb{N})$ has a cluster-point in Δ.

If β does not win a play of the $\mathcal{G}_D(\Delta)$-game then we declare that α *wins* that play of the $\mathcal{G}_D(\Delta)$-game. As with the earlier games we need to define the notion of a strategy.

By a *strategy* σ for the player β we mean a 'rule' that specifies each move of the player β in every possible situation. More precisely, a strategy $\sigma := (\sigma_n : n \in \mathbb{N})$ for β is an inductively defined sequence of functions. The domain of σ_1 is the sequence of length zero, denoted by $\emptyset$. That is, $\mathrm{Dom}(\sigma_1) = \{\emptyset\}$ and $\sigma_1(\emptyset)$ is an open neighbourhood of Δ. If $\sigma_1, \sigma_2, \ldots, \sigma_k$ have been defined then the domain of σ_{k+1} is:

$$\{((x_1, y_1), \ldots, (x_k, y_k)) \in (D \times D)^k : ((x_1, y_1), \ldots (x_{k-1}, y_{k-1})) \in \mathrm{Dom}(\sigma_k) \text{ and } (x_k, y_k) \in \sigma_k((x_1, y_1), \ldots, (x_{k-1}, y_{k-1}))\}.$$

For each $((x_1, y_1), \ldots, (x_k, y_k)) \in \mathrm{Dom}(\sigma_{k+1})$, $\sigma_{k+1}((x_1, y_1), \ldots, (x_k, y_k))$ is an open neighbourhood of Δ.

A *partial σ-play* is a finite sequence $((x_1, y_1), \ldots, (x_n, y_n))$ such that

$$((x_1, y_1), \ldots, (x_n, y_n)) \in \text{Dom}(\sigma_{n+1}).$$

A *σ-play* is an infinite sequence $((x_n, y_n) : n \in \mathbb{N})$ such that $((x_1, y_1), \ldots, (x_n, y_n))$ is a partial σ-play for each $n \in \mathbb{N}$, .

A strategy $\sigma := (\sigma_n : n \in \mathbb{N})$ for the player β is called a *winning strategy* if each play of the form: $\big(((x_n, y_n), \sigma_n((x_1, y_1), \ldots, (x_{n-1}, y_{n-1}))) : n \in \mathbb{N}\big)$ is won by β.

Suppose that (X, τ) and (Y, τ') are topological spaces and $f : X \times Y \to \mathbb{R}$. If $0 < \varepsilon$ and $(x_0, y_0) \in X \times Y$ then we say that f is *ε-jointly continuous at* $(x_0, y_0) \in X \times Y$ if there exist open neighbourhoods U of x_0 and V of y_0 such that $\text{diam}[f(U \times V)] \leq \varepsilon$.

To prove the main theorem in the section it is helpful to first consider the following preliminary result regarding ε-continuity.

Lemma 1.3.1. *Suppose that (X, τ) and (Y, τ') are topological spaces and $f : X \times Y \to \mathbb{R}$ is a separately continuous function. Suppose also that D is a dense subset of (Y, τ') and $0 < \varepsilon$. If $(x_0, y_0) \in X \times Y$ and f is not ε-continuous at (x_0, y_0) then, for every pair of open neighbourhoods U of x_0 and V of y_0 there exists points $(d, d') \in (D \times D) \cap (V \times V)$ and $x \in U$ such that $\varepsilon/3 < |f(x, d) - f(x, d')|$.*

Proof. Let us prove the contrapositive statement. Suppose that U_0 and V_0 are open neighbourhoods of x_0 and y_0 respectively, such that, for every $(d, d') \in (D \times D) \cap (V_0 \times V_0)$ and every $x \in U_0$, $|f(x, d) - f(x, d')| \leq \varepsilon/3$. Firstly, note that since f is separately continuous

$$|f(x, y) - f(x, y')| \leq \varepsilon/3 \quad \text{for all } x \in U_0 \text{ and all } (y, y') \in V_0 \times V_0.$$

Next, note that by possibly making U_0 smaller, if necessary, we may assume that

$$|f(x, y_0) - f(x', y_0)| < \varepsilon/3 \quad \text{for all } x, x' \in U_0.$$

We claim that $\text{diam}[f(U_0 \times V_0)] \leq \varepsilon$. To this end, let $(x, y), (x', y') \in U_0 \times V_0$. Then,

$$\begin{aligned} |f(x, y) - f(x', y')| &\leq |f(x, y) - f(x, y_0)| + |f(x, y_0) - f(x', y_0)| + |f(x', y_0) - f(x', y')| \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Therefore, $|f(x, y) - f(x', y')| \leq \varepsilon$; which shows that f is ε-jointly continuous at (x_0, y_0). □

Theorem 1.3.2 ([5, Theorem 2])**.** *Suppose that (Y, τ') is a compact topological space such that for some dense subset D of (Y, τ'), the player β has a winning strategy in the $\mathcal{G}_D(\Delta)$-game played on $Y \times Y$, then (Y, τ') is a co-Namioka space.*

Proof. Let (X, τ) be a Baire space and $f : X \times Y \to \mathbb{R}$ be a separately continuous function. Fix $0 < \varepsilon$ and consider the set

$$O_\varepsilon := \{x \in X : f \text{ is } \varepsilon\text{-jointly continuous at each point of } \{x\} \times Y\}.$$

It follows from the definition of ε-joint continuity and a simple compactness argument that the set O_ε is an open subset of (X, τ). We claim that O_ε is dense in (X, τ). So, for the purpose of obtaining a contradiction, let us suppose that there exists a nonempty open subset W of X such that $O_\varepsilon \cap W = \varnothing$. Let $\sigma := (\sigma_n : n \in \mathbb{N})$ be a winning strategy for the player β in the $\mathcal{G}_D(\Delta)$-game played on $Y \times Y$.

We shall inductively define a (necessarily non-winning) strategy $t := (t_n : n \in \mathbb{N})$ for the player β in the Choquet-game played on (X, τ).

Step 1. Let $x \in W$. Then $x \notin O_\varepsilon$. Therefore, there exists a point $y \in Y$ such that f is not ε-jointly continuous at (x, y). Let V be an open neighbourhood of y such that $(y, y) \in V \times V \subseteq \sigma_1(\emptyset)$. By Lemma 1.3.1 there exist a point $(y_\emptyset, y'_\emptyset) \in (D \times D) \cap (V \times V)$ and a point $x_1 \in W$ such that $\varepsilon/3 < |f(x_1, y_\emptyset) - f(x_1, y'_\emptyset)|$. Let

$$t_1(\emptyset) := \{x \in W : \varepsilon/3 < |f(x, y_\emptyset) - f(x, y'_\emptyset)|\}.$$

Note that $t_1(\emptyset)$ is open and nonempty, as $x_1 \in t_1(\emptyset)$. Furthermore, $(y_\emptyset, y'_\emptyset) \in (D \times D) \cap \sigma_1(\emptyset)$.

Step 2. Suppose that A_1 is a nonempty open subset of $t_1(\emptyset)$. We may think of A_1 as the first move of the player α in the $Ch(X)$-game.

Let $x \in A_1$. Then $x \notin O_\varepsilon$. Therefore, there exists a point $y \in Y$ such that f is not ε-jointly continuous at (x, y). Since $(y_\emptyset, y'_\emptyset) \in (D \times D) \cap \sigma_1(\emptyset)$, $\sigma_2((y_\emptyset, y'_\emptyset))$ is defined.

Let V be an open neighbourhood of y such that $(y, y) \in V \times V \subseteq \sigma_2((y_\emptyset, y'_\emptyset))$. By Lemma 1.3.1 there exist a point $(y_{(A_1)}, y'_{(A_1)}) \in (D \times D) \cap (V \times V)$ and a point $x_2 \in A_1$ such that $\varepsilon/3 < |f(x_2, y_{(A_1)}) - f(x_2, y'_{(A_1)})|$. Define

$$t_2(A_1) := \{x \in A_1 : \varepsilon/3 < |f(x, y_{(A_1)}) - f(x, y'_{(A_1)})|\}.$$

Note that $t_2(A_1)$ is open and nonempty. Furthermore, $(y_{(A_1)}, y'_{(A_1)}) \in (D \times D) \cap \sigma_2((y_\emptyset, y'_\emptyset))$.

Now, let $n \in \mathbb{N} \setminus \{1\}$ and suppose that the points $(y_{(A_1,\ldots,A_{j-1})}, y'_{(A_1,\ldots,A_{j-1})}) \in D \times D$ and t_j have been defined for each partial t-play $(A_1, \ldots, A_{j-1})$ of length $(j-1)$ with $2 \le j \le n$ so that:

(i) $\Big((y_\emptyset, y'_\emptyset), \ldots, (y_{(A_1,\ldots,A_{j-2})}, y'_{(A_1,\ldots,A_{j-2})})\Big)$ is a partial σ-play;

(ii) $(y_{(A_1,\ldots,A_{j-1})}, y'_{(A_1,\ldots,A_{j-1})}) \in (D \times D) \cap \sigma_j((y_\emptyset, y'_\emptyset), \ldots, (y_{(A_1,\ldots,A_{j-2})}, y'_{(A_1,\ldots,A_{j-2})}))$ and

(iii) $t_j(A_1, \ldots, A_{j-1}) := \{x \in A_{j-1} : \varepsilon/3 < |f(x, y_{(A_1,\ldots,A_{j-1})}) - f(x, y'_{(A_1,\ldots,A_{j-1})})|\}$.

Step $n+1$. Let $(A_1, \ldots, A_n)$ be a partial t-play of length n. Let $x \in A_n$. Then $x \notin O_\varepsilon$. Therefore, there exists a point $y \in Y$ such that f is not ε-jointly continuous at (x, y).

By (i) and (ii) above $\Big((y_\emptyset, y'_\emptyset), \ldots, (y_{(A_1,\ldots,A_{n-1})}, y'_{(A_1,\ldots,A_{n-1})})\Big)$ is a partial σ-play. Therefore,

$$\sigma_{n+1}\Big((y_\emptyset, y'_\emptyset), \ldots, (y_{(A_1,\ldots,A_{n-1})}, y'_{(A_1,\ldots,A_{n-1})})\Big) \quad \text{is defined.}$$

Let V be an open neighbourhood of y such that

$$(y, y) \in V \times V \subseteq \sigma_{n+1}\Big((y_\emptyset, y'_\emptyset), \ldots, (y_{(A_1,\ldots,A_{n-1})}, y'_{(A_1,\ldots,A_{n-1})})\Big).$$

By Lemma 1.3.1 there exist a point $(y_{(A_1,\ldots,A_n)}, y'_{(A_1,\ldots,A_n)}) \in (D \times D) \cap (V \times V)$ and a point $x_{n+1} \in W$ such that $\varepsilon/3 < |f(x_{n+1}, y_{(A_1,\ldots,A_n)}) - f(x_{n+1}, y'_{(A_1,\ldots,A_n)})|$. Let

$$t_{n+1}(A_1, \ldots, A_n) := \{x \in A_n : \varepsilon/3 < |f(x, y_{(A_1,\ldots,A_n)}) - f(x, y'_{(A_1,\ldots,A_n)})|\}.$$

Note that $t_{n+1}(A_1, \ldots, A_n)$ is open and nonempty as $x_{n+1} \in t_{n+1}(A_1, \ldots, A_n)$. Furthermore, $(y_{(A_1,\ldots,A_n)}, y'_{(A_1,\ldots,A_n)}) \in (D \times D) \cap \sigma_{n+1}((y_\emptyset, y'_\emptyset), \ldots, (y_{(A_1,\ldots,A_{n-1})}, y'_{(A_1,\ldots,A_{n-1})}))$.

This completes the definition of the strategy $t := (t_n : n \in \mathbb{N})$. Now, since (X, τ) is a Baire space t is not a winning strategy for the player β, see Theorem 1.1.5. Hence there is a t-play $(A_n : n \in \mathbb{N})$ where the player α wins, i.e., $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$. Let $x_\infty \in \bigcap_{n\in\mathbb{N}} A_n$. Note also

$$\Big((y_\emptyset, y'_\emptyset), (y_{(A_1)}, y'_{(A_1)}), \ldots, (y_{(A_1,\ldots,A_n)}, y'_{(A_1,\ldots,A_n)}), \ldots\Big)$$

is a σ-play of the $\mathcal{G}_D(\Delta)$-game. Hence, the sequence $((y_{(A_1,\ldots,A_n)}, y'_{(A_1,\ldots,A_n)}) : n \in \mathbb{N})$ has a cluster-point $(y_\infty, y_\infty) \in \Delta$. Let $V := \{y \in Y : |f(x_\infty, y) - f(x_\infty, y_\infty)| < \varepsilon/6\}$. Then:

(i) V is an open subset of Y, since f is a separately continuous function;

(ii) $(y_\infty, y_\infty) \in V \times V$ and

(iii) $|f(x_\infty, y) - f(x_\infty, y')| < \varepsilon/3$ for all $(y, y') \in V \times V$, by the triangle inequality.

However, for n large enough, $(y_{(A_1,\ldots,A_n)}, y'_{(A_1,\ldots,A_n)}) \in V \times V$; which is impossible since $x_\infty \in t_{n+1}(A_1, \ldots, A_n)$ and so $\varepsilon/3 < |f(x_\infty, y_{(A_1,\ldots,A_n)}) - f(x_\infty, y'_{(A_1,\ldots,A_n)})|$. Therefore, our assumption that O_ε is not dense, is false, i.e., O_ε is dense in (X, τ).

It now only remains to see that f is jointly continuous at each point of $(\bigcap_{n\in\mathbb{N}} O_{1/n}) \times Y$. $\square$

We now show that there are some non-trivial spaces where the player β has a winning strategy in the $\mathcal{G}_D(\Delta)$-game.

Theorem 1.3.3 ([5, Theorem 3]). *Let (Y, τ') be a compact topological space, D be a dense subset of (Y, τ') and $\mathscr{U}$ be an open cover of $(Y \times Y) \setminus \Delta$ such that:*

(i) *for every $U \in \mathscr{U}$, $\overline{U} \cap \Delta = \varnothing$;*

(ii) *for every $(x, y) \in D \times D$, $\{U \in \mathscr{U} : (x, y) \in U\}$ is at most countable*

then the player β has a winning strategy in the $\mathcal{G}_D(\Delta)$-game played on $Y \times Y$. In particular, (Y, τ') is a co-Namioka space.

Proof. For $(x, y) \in D \times D$, let $\{U_n(x, y) : n \in \mathbb{N}\}$ be an enumeration of all $U \in \mathscr{U}$ with $(x, y) \in U$. The winning strategy $\sigma := (\sigma_n : n \in \mathbb{N})$ is inductively defined by, $\sigma_1(\emptyset) := Y \times Y$. If $n \in \mathbb{N}$ and $((x_1, y_1), \ldots, (x_n, y_n))$ is a partial σ-play of length n then

$$\sigma_{n+1}((x_1, y_1), \ldots, (x_n, y_n)) := (Y \times Y) \setminus \bigcup\{\overline{U_i(x_j, y_j)} : 1 \leq i \leq n \text{ and } 1 \leq i \leq n\}.$$

This completes the definition of $\sigma := (\sigma_n : n \in \mathbb{N})$.

We now show that σ is a winning strategy for the player β. To this end, let $((x_n, y_n) : n \in \mathbb{N})$ be a σ-play of the $\mathcal{G}_D(\Delta)$-game played on $Y \times Y$. Let (x, y) be a cluster-point of

$((x_n, y_n) : n \in \mathbb{N})$. We claim that $(x, y) \in \Delta$. Indeed, if $(x, y) \notin \Delta$ then there exists a $U \in \mathscr{U}$ such that $(x, y) \in U$. Since (x, y) is a cluster-point of the sequence $((x_n, y_n) : n \in \mathbb{N})$ there exists a $k \in \mathbb{N}$ such that $(x_k, y_k) \in U$. Therefore, there exists an $m \in \mathbb{N}$ such that $U = U_m(x_k, y_k)$. Let $N := \max\{m, k\}$. Then $(x_n, y_n) \notin U_m(x_k, y_k) = U$ for all $N < n$. So (x, y) cannot be a cluster-point of $((x_n, y_n) : n \in \mathbb{N})$ after all. Therefore, $(x, y) \in \Delta$. This shows that σ is indeed a winning strategy for the player β. □

We shall call a compact Hausdorff space (Y, τ') a *Valdivia compact* if there exists a homeomorphism $h : Y \to Z$ from Y onto a topological space (Z, τ) and there exists an index set I such that $Z \subseteq [0, 1]^I$ and $\Sigma(I) \cap Z$ is dense in Z where, $\Sigma(I) := \{x \in [0, 1]^I : \{i \in I : x(i) \neq 0\}$ is at most countable$\}$.

Corollary 1.3.4 ([5, 10])**.** *Every Valdivia compact space is a co-Namioka space.*

Proof. Let (Y, τ') be a Valdivia compact space. We may assume, after possibly considering a homeomorphic copy of (Y, τ'), rather than the original space (Y, τ'), that there exists an index set I such that $Y \subseteq [0, 1]^I$ and $\Sigma(I) \cap Y$ is dense in Y where, $\Sigma(I) := \{x \in [0, 1]^I : \{i \in I : x(i) \neq 0\}$ is at most countable$\}$.

Let $D := \Sigma(I) \cap Y$. For each $(i, n) \in I \times \mathbb{N}$, let

$$U_{(i,n)} := \{(x, y) \in Y \times Y : 1/n < |y(i) - x(i)|\}$$

then let $\mathscr{U} := \{U_{(i,n)} : (i, n) \in I \times \mathbb{N}\}$. If $(x, y) \in D \times D$ then $\{U \in \mathscr{U} : (x, y) \in U\}$ is at most countable. Furthermore, $\overline{U} \cap \Delta = \varnothing$ for each $U \in \mathscr{U}$. Hence, the cover $\mathscr{U}$ satisfies the hypotheses of Theorem 1.3.3. Therefore, (Y, τ') is a co-Namioka space. □

1.4 Fragmentable spaces and games

In this section we will review the relationship between fragmentability and separate and joint continuity. We shall start with the following slight generalisation of Theorem 1.6.23, which is phrased in terms bitopological spaces. Recall that an ordered triple (X, τ, τ') is called a *bitopological space* if both (X, τ) and (X, τ') are topological spaces.

Theorem 1.4.1 ([30, 31])**.** *Suppose that (X, τ) is a topological space, (Y, τ', τ'') is a bitopological space and $f : X \to Y$ is a τ'-quasicontinuous function. If (Y, τ') is fragmented by a metric d whose topology on Y is at least as strong as τ'' then, f is τ''-continuous at the points of a residual subset of (X, τ).*

Proof. The proof of this is left to the reader. However, the proof only requires a slight modification of the proof of Theorem 1.6.23. □

The next result reveals our interest in fragmentability in bitopological spaces.

Corollary 1.4.2. *Let (Y, τ') be a compact Hausdorff topological space. If $C_p(Y)$ is fragmented by a metric d whose topology on $C(Y)$ is at least as strong as the norm topology on $C(Y)$, then (Y, τ') is a co-Namioka space.*

Proof. This follows directly from Theorem 1.4.1 with $\tau' = \tau_p(Y)$ and τ'', the topology on $C(Y)$ generated by the supremum norm on $C(Y)$. $\square$

To progress further in our studies of fragmentable spaces we will need to consider a new topological game.

As with all the earlier the games, this game involves two players, but unlike the previous games, the players are named Σ and Ω. The game is played on is a fixed bitopological space (X, τ, τ').

The name of the game is the *fragmenting game* and is denoted by, $\mathcal{G}(\tau, \tau')$.

We now describe how to "play" the $\mathcal{G}(\tau, \tau')$-game. The player labeled Σ starts the game, (every time). For his/her first move the player Σ must select nonempty subset A_1 of X. Next, Ω gets a turn. For Ω's first move he/she must select a nonempty subset B_1 of A_1, which is relatively τ-open in B_1. This ends the first round of the game.

In the second round, Σ goes first again and selects a nonempty subset $A_2 \subseteq B_1$. Player Ω then gets to respond by choosing a nonempty subset B_2 of A_2, which is relatively τ-open in A_2. This ends the second round of the game.

In general, after Σ and Ω have played the first n-rounds of the $\mathcal{G}(\tau, \tau')$-game, Σ will have selected nonempty subsets $A_1, A_2, \ldots, A_n$ of X and Ω will have selected nonempty subsets $A_1, A_2, \ldots, A_n$ of X such that

$$B_n \subseteq A_n \subseteq B_{n-1} \subseteq A_{n-1} \subseteq \cdots \subseteq B_2 \subseteq A_2 \subseteq B_1 \subseteq A_1$$

and B_j is relatively τ-open in A_j for all $1 \leq j \leq n$. At the start of the $(n+1)$-round of the game, Σ goes first (again!) and selects nonempty subset A_{n+1} of B_n. As with the previous n-rounds, the player Ω gets to respond to this move by selecting a nonempty subset B_{n+1} of A_{n+1}, which is relatively τ-open in A_{n+1}.

"Plays" and "partial plays" etc. are defined in an analogous way to our earlier games.

As with any game, we need to specify a rule to determine who wins. We shall declare that Ω *wins* a play $\big((A_k, B_k) : k \in \mathbb{N}\big)$ of the $\mathcal{G}(\tau, \tau')$-game if:

(i) $\bigcap_{k\in\mathbb{N}} A_k = \bigcap_{k\in\mathbb{N}} B_k = \varnothing$, or

(ii) $\bigcap_{k\in\mathbb{N}} A_k = \bigcap_{k\in\mathbb{N}} B_k = \{x\}$ for some $x \in X$ and for every $U \in \tau'$ with $x \in U$ there exists a $k \in \mathbb{N}$ such that $A_k \subseteq U$.

If Ω does not win a play of the $\mathcal{G}(\tau, \tau')$-game then we declare that Σ *wins* that play of the $\mathcal{G}(\tau, \tau')$-game. So every play is won by either Ω or Σ and no play is won by both players.

Strategies and t-plays/partial t-plays etc. are defined in an analogous way to our earlier games.

In order to provide a game characterisation of fragmentability we will need to introduce and intermediate notion. Let (X, τ) be topological space. Then we call $\mathcal{P} \subseteq 2^X \setminus \{\varnothing\}$ a *partial exhaustive partition of* X if:

(i) $\bigcup_{P\in\mathcal{P}} P \in \tau$;

(ii) the members of $\mathcal{P}$ are disjoint;

(iii) for every nonempty subset A of $\bigcup_{P\in\mathcal{P}} P$ there exists a $P \in \mathcal{P}$ such that $A \cap P$ is nonempty and $A \cap P$ is relatively τ-open in A.

If $\bigcup_{P\in\mathcal{P}} P = X$, then we simply call $\mathcal{P}$ an *exhaustive partition of* X.

We shall need the following technical lemma.

Lemma 1.4.3. *If $\mathcal{P}$ is a partial exhaustive partition of a topological space (X,τ) and $X \setminus \bigcup_{P\in\mathcal{P}} P \neq \varnothing$ and W is a nonempty subset of $X \setminus \bigcup_{P\in\mathcal{P}} P$ that is relatively τ-open in $X \setminus \bigcup_{P\in\mathcal{P}} P$, then $\mathcal{P} \cup \{W\}$ is also a partial exhaustive partition of X.*

Proof. Since W is a relatively τ-open subset of $X \setminus \bigcup_{P\in\mathcal{P}} P$, there exists a τ-open subset U of X such that $W = (X \setminus \bigcup_{P\in\mathcal{P}} P) \cap U$. Therefore,

$$\bigcup_{P\in\mathcal{P}\cup\{W\}} P = (\bigcup_{P\in\mathcal{P}} P) \cup W = (\bigcup_{P\in\mathcal{P}} P) \cup U.$$

Hence, as the union of two open sets, $\bigcup_{P\in\mathcal{P}\cup\{W\}} P$ is τ-open in X. Furthermore, it is easy to see that the members of $\mathcal{P} \cup \{W\}$ are pairwise disjoint. Next, suppose that A is a nonempty subset of $\bigcup_{P\in\mathcal{P}\cup\{W\}} P$. Let $V := \bigcup_{P\in\mathcal{P}} P$. If $V \cap A \neq \varnothing$ then there exists a member $P_0 \in \mathcal{P}$ such that $(V \cap A) \cap P_0$ is nonempty and $(V \cap A) \cap P_0$ is relatively τ-open in $(A \cap V)$. Since $A \cap V$ is relatively τ-open in A, $(V \cap A) \cap P_0$ is relatively τ-open in A. Moreover, since $P_0 \subseteq V$,

$$A \cap P_0 = (A \cap V) \cap P_0 \neq \varnothing$$

and so $A \cap P_0$ is a nonempty and relatively τ-open in A. If $V \cap A = \varnothing$ then $A \subseteq W$ and so $A \cap W = A$; which is of course relatively τ-open in A. Hence, in both cases there exists a member $P \in \mathcal{P} \cup \{W\}$ such that $A \cap P \neq \varnothing$ and $A \cap P$ is relatively τ-open in A. This completes the proof of the lemma. □

Theorem 1.4.4 ([32])**.** *Let (X,τ,τ') be a bitopological space. Then (X,τ) is fragmented by a metric d whose topology on X is at least as strong as τ' if, and only if, the player Ω has a winning strategy in the $\mathcal{G}(\tau,\tau')$-game played on (X,τ,τ').*

Proof. Suppose that (X,τ) is fragmented by a metric d whose topology on X is at least as strong as τ'. Then the strategy $s := (s_n \in \mathbb{N})$ for the player Ω is clear.

Step 1. If A_1 is a nonempty open subset of X then $s_1(A_1)$ is defined to be any nonempty subset of A_1 that is relatively τ-open in A_1 and $d-\text{diam}[s_1(A_1)] < 1$. Such a set is guaranteed by the fact that (X,τ) is fragmented by the metric d.

Now, let $n \in \mathbb{N}$ and suppose that s_j has been defined for each partial s-play $(A_1,\ldots,A_j)$ of length j with $1 \leq j \leq n$ so that: $s_j(A_1,\ldots,A_j)$ is any nonempty subset of A_j, $s_j(A_1,\ldots,A_j)$ is relatively τ-open in A_j and $d-\text{diam}[s_j(A_1,\ldots,A_j)] < 1/j$.

Step $n+1$. Let $(A_1,\ldots,A_{n+1})$ be a partial s-play of length $n+1$. Then A_{n+1} is a nonempty subset of X. Let $s_{n+1}(A_1,\ldots,A_{n+1})$ be defined to be any nonempty subset of A_{n+1} that is relatively τ-open in A_{n+1} and $d-\text{diam}[s_{n+1}(A_1,\ldots,A_{n+1})] < 1/(n+1)$. Such a set is guaranteed by the fact that (X,τ) is fragmented by the metric d.

This completes the definition of $s := (s_n : n \in \mathbb{N})$. To see that s is a winning strategy for the player Ω, consider the following: Let $(A_n : n \in \mathbb{N})$ be an arbitrary s-play. If $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$, then Ω wins this play. If $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$ then $\bigcap_{n\in\mathbb{N}} A_n = \{x\}$ for some $x \in X$, since $\text{diam}[\bigcap_{k\in\mathbb{N}} A_k] \leq \text{diam}[A_{n+1}] < 1/n$ for all $n \in \mathbb{N}$ and so $\text{diam}[\bigcap_{k\in\mathbb{N}} A_k] = 0$. Furthermore, if $U \in \tau'$ and $x \in U$ then there exists an $n \in \mathbb{N}$ such that

$$B_d(x, 1/n) := \{y \in X : d(y, x) < 1/n\} \subseteq U,$$

since the d-topology at x is at least as strong as the τ'-topology at x. Therefore,

$$x \in A_{n+1} \subseteq B_d(x, 1/n) \subseteq U.$$

Thus, the player Ω, wins the play $(A_n : n \in \mathbb{N})$ in this case too.

We now consider the converse.

Suppose the player Ω has a winning strategy $s := (s_n : n \in \mathbb{N})$ in the $\mathcal{G}(\tau, \tau')$-game played on (X, τ, τ'). We need to firstly, construct a metric d on X, then show that the d-topology on X is at least as strong as the τ'-topology on X and then finally, show that the metric d actually fragments the topological space (X, τ).

Part I: For each $n \in \mathbb{N}$, let P_n denote the set of all partial s-plays of length n, and for purely notational reasons, let us set $\Lambda_0 = \{\emptyset\}$. We shall inductively define a sequence $(\Lambda_n : n \in \mathbb{N})$ of subsets such that the following properties are fulfilled. For each $n \in \mathbb{N}$:

(a_n) $\Lambda_n \subseteq P_n$ and $s_n(p) \cap s_n(p') = \varnothing$ for every distinct $p, p' \in \Lambda_n$;

(b_n) $\{s_n(p) : p \in \Lambda_n\}$ is an exhaustive partition of X;

(c_n) for each $p \in \Lambda_n$, $p|_j \in \Lambda_j$ for all $0 \leq j < n$.

Step 1. Let Λ_1 be a maximal subset of P_1 such that (a_1) and (c_1) are satisfied and $\{s_1(p) : p \in \Lambda_1\}$ is partial exhaustive partition of X. By Zorn's Lemma such a maximal subset exists.

We claim that $\{s_1(p) : p \in \Lambda_1\}$ is an exhaustive partition of X. To see this, suppose that $A := X \setminus \{s_1(p) : p \in \Lambda_1\} \neq \varnothing$. Note that (A) is an s-play of length 1. Let $\Lambda^* := \Lambda_1 \cup \{(A)\}$. Then Λ^* satisfies (a_1) and (c_1). Furthermore, by Lemma 1.4.3, $\{s_1(p) : p \in \Lambda^*\}$ is a partial exhaustive partition of X, since $s_1(A)$ is a relatively τ-open subset of $X \setminus \{s_1(p) : p \in \Lambda_1\}$. However, this contradicts the maximality of Λ_1. Hence, $\{s_1(p) : p \in \Lambda_1\}$ must indeed be an exhaustive partition of X.

Let $n \in \mathbb{N}$, and suppose the subsets Λ_k satisfying the Properties (a_k), (b_k) and (c_k) have been defined for each $1 \leq k \leq n$.

Step $n+1$. Let Λ_{n+1} be a maximal subset of P_{n+1} such that (a_{n+1}) and (c_{n+1}) are satisfied and $\{s_{n+1}(p) : p \in \Lambda_{n+1}\}$ is partial exhaustive partition of X. By Zorn's Lemma such a maximal subset exists.

We claim that $\{s_{n+1}(p) : p \in \Lambda_{n+1}\}$ is an exhaustive partition of X. If not, then

$$A := X \setminus \bigcup\{s_{n+1}(p) : p \in \Lambda_{n+1}\} \neq \varnothing.$$

Since $\{s_n(p) : p \in \Lambda_n\}$ is an exhaustive partition of X there exists a $p \in \Lambda_n$ such that $A' := s_n(p) \cap A$ is nonempty and relatively τ-open in A. Note also that (p, A') is a partial s-play of length $n+1$. Let $\Lambda^* := \Lambda_{n+1} \cup \{(p, A')\}$. Then Λ^* satisfies (a_{n+1}), obviously, and (c_{n+1}). Furthermore, since $s_{n+1}(p, A')$ is a relatively τ-open subset of A' and A' is relatively τ-open subset of A, $s_{n+1}(p, A')$ is a relatively τ-open subset of $A = X \setminus \bigcup\{s_{n+1}(p) : p \in \Lambda_{n+1}\}$.

Therefore, by Lemma 1.4.3, $\{s_{n+1}(p) : p \in \Lambda^*\}$ is a partial exhaustive partition of X. However, this contradicts the maximality of Λ_{n+1}. Hence, $\{s_{n+1}(p) : p \in \Lambda_{n+1}\}$ must be an exhaustive partition of X. This completes the induction.

We now claim that for each $x \in X$ there exists a unique s-play $p := (A_n : n \in \mathbb{N})$ such that $x \in \bigcap_{n\in\mathbb{N}} A_n$ and $p|_n \in \Lambda_n$ for all $n \in \mathbb{N}$.

We first show *existence*. To this end, let us consider $x \in X$ and $n \in \mathbb{N}$. Then, by the Properties (a_n) and Property (b_n), there exists a unique $p_n \in \Lambda_n$ such that $x \in s_n(p_n)$.

We show that if $1 \leq n < m$ then $p_m|_n = p_n$. To see this, first note that $p_m|_n \in \Lambda_n$, by Property (c_m), and secondly, that $x \in s_n(p_m|_n) \cap s_n(p_n)$. Therefore, by Property (a_n), it must be the case that $p_m|_n = p_n$. Thus, p_m is a continuation of the partial s-play. Let $p := (A_n^n : n \in \mathbb{N})$, where for each $n \in \mathbb{N}$, $p_n := (A_1^n, \ldots, A_n^n)$. Clearly, $x \in \bigcap_{n\in\mathbb{N}} A_n^n$ as

$$x \in s_n(p_n) = s_n(A_1^n, \ldots, A_n^n) \subseteq A_n^n \quad \text{for all } n \in \mathbb{N}.$$

We now demonstrate that p is an s-play and that $p|_n = p_n \in \Lambda_n$ for all $n \in \mathbb{N}$. To this end, let $n \in \mathbb{N}$. Then by above, for each $1 \leq j < n$, $p_j = p_n|_j$. Therefore, for each $1 \leq j < n$, $A_j^j = A_j^n$. Thus,

$$p|_n = (A_1^1, \ldots A_n^n) = (A_1^n, \ldots, A_n^n) = p_n \in \Lambda_n.$$

Furthermore, since $p_{n+1} = (A_1^{n+1}, \ldots, A_{n+1}^{n+1}) \in \Lambda_{n+1} \subseteq P_{n+1}$,

$$A_{n+1}^{n+1} \subseteq s_n(A_1^{n+1}, \ldots, A_n^{n+1}). \qquad (*)$$

Now again, by above, for each $1 \leq j \leq n$, $p_{n+1}|_j = p_j$. Therefore, for each $1 \leq j \leq n$, $A_j^{n+1} = A_j^j$. Substituting this into Equation $(*)$ we get that

$$A_{n+1}^{n+1} \subseteq s_n(A_1^1, \ldots, A_n^n).$$

This show that p is an s-play.

We now show *uniqueness*. Suppose that $p := (A_n : n \in \mathbb{N})$ and $p' := (A'_n : n \in \mathbb{N})$ are s-plays such that:

(i) $x \in \bigcap_{n\in\mathbb{N}} A_n$ and $x \in \bigcap_{n\in\mathbb{N}} A'_n$ and

(ii) $p|_n \in \Lambda_n$ and $p'|_n \in \Lambda_n$ for every $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$. Since $x \in s_n(p|_n) \cap s_n(p'|_n)$, it follows from the second part of Property (a_n), that that $p|_n = p'|_n$. In particular, $A_n := A'_n$. This shows the uniqueness.

For each $x \in X$, let $p(x)$ denote the unique s-play $(A_n : n \in \mathbb{N})$ such that $x \in \bigcap_{n\in\mathbb{N}} A_n$ and $p|_n \in \Lambda_n$ for all $n \in \mathbb{N}$.

We now define a metric d on X as follows. Suppose that $x, y \in X$, $p(x) := (A_n : n \in \mathbb{N})$ and $p(y) := (A'_n : n \in \mathbb{N})$. Then

$$d(x,y) := \begin{cases} 0 & \text{if } p(x) = p(y) \\ 1/n & \text{if } p(x) \neq p(y) \end{cases} \qquad \text{where, } n := \min\{i \in \mathbb{N} : A_i \neq A'_i\}.$$

Clearly, $0 \leq d(x,y)$ for all $x, y \in X$, and $d(x,x) = 0$ for all $x \in X$ follows from the uniqueness of $p(x)$ and $p(y)$. On the other hand, if $d(x,y) = 0$ then $A_k = A'_k$ for all $k \in \mathbb{N}$. Furthermore, since both $(A_n : n \in \mathbb{N})$ and $(A'_n : n \in \mathbb{N})$ are s-plays and $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$ and $\bigcap_{n\in\mathbb{N}} A'_n \neq \varnothing$, it must be the case that

$$\{x\} = \bigcap_{k\in\mathbb{N}} A_k = \bigcap_{k\in\mathbb{N}} A'_k = \{y\}.$$

Therefore, $x = y$. It follows directly from the definition of d that it is symmetric, i.e., $d(x,y) = d(y,x)$ for all $x, y \in X$. So it remains to show the triangle inequality. Let $x, y, z \in X$. We will show that $d(x,z) \leq d(x,y) + d(y,z)$. Let $p(x) := (A_n : n \in \mathbb{N})$, $p(y) := (A'_n : n \in \mathbb{N})$ and $p(z) := (A''_n : n \in \mathbb{N})$. Set $n_{xz} := \min\{i \in \mathbb{N} : A_i \neq A''_i\}$, $n_{xy} := \min\{i \in \mathbb{N} : A_i \neq A'_i\}$ and $n_{yz} := \min\{i \in \mathbb{N} : A'_i \neq A''_i\}$. If $i < \min\{n_{xy}, n_{yz}\}$ then $A_i = A'_i$ and $A'_i = A''_i$. That is, $A_i = A''_i$. Therefore,

$$\{i \in \mathbb{N} : A_i \neq A''_i\} \subseteq \{i \in \mathbb{N} : \min\{n_{xy}, n_{yz}\} \leq i\}.$$

Hence, $\min\{n_{xy}, n_{yz}\} \leq n_{xz}$. Thus, either

$$d(x,z) = 1/n_{xz} \leq 1/n_{xy} = d(x,y) \quad \text{or} \quad d(x,z) = 1/n_{xz} \leq 1/n_{yz} = d(y,z) \quad \text{(or both)}.$$

It now follows that $d(x,z) \leq d(x,y) + d(y,z)$.

Part II: The next thing we have to show is that the d-topology on X is at least as strong as the τ'-topology on X. To this end, let $x \in X$ and let $U \in \tau'$ be such that $x \in U$. Since $p(x) := (A_n : n \in \mathbb{N})$ is an s-play and $\bigcap_{n\in\mathbb{N}} A_n = \{x\} \subseteq U$ there exists an $n \in \mathbb{N}$ such that $A_n \subseteq U$. We claim that $B_d(x, 1/n) \subseteq U$. To see this, consider $y \in B_d(x, 1/n)$. Then $p(y) := (A'_n : n \in \mathbb{N})$ and $d(x,y) < 1/n$. If $x = y$ then $y \in U$. So let us suppose that $x \neq y$. Then $n < 1/d(x,y)$, i.e., $n < \min\{k \in \mathbb{N} : A_k \neq A'_k\}$. Thus, $A_n = A'_n$ and so

$$y \in \bigcap_{k\in\mathbb{N}} A'_k \subseteq A'_n = A_n \subseteq U.$$

This shows that $B_d(x, 1/n) \subseteq U$.

Part III: Thus, it remains to show that d fragments (X, τ). So with this in mind, let us consider an arbitrary nonempty subset C of X and an arbitrary positive real number ε. Choose $n \in \mathbb{N}$ such that $1/n < \varepsilon$. Then, since $\{s_n(p) : p \in \Lambda_n\}$ is an exhaustive partition of (X, τ), [recall Property (b_n)], there exists a $q_n \in \Lambda_n$ such that $C \cap s_n(q_n)$ is nonempty and relatively τ-open in C. Let $x, y \in C \cap s_n(q_n)$. We assert that $d(x,y) < 1/n$. To justify this assertion we need to examine the definition of d. To this end, let $p(x) := (A_k : k \in \mathbb{N})$ and $p(y) := (A'_k : k \in \mathbb{N})$ be the unique s-plays such that $x \in \bigcap_{k\in\mathbb{N}} A_k$ and $(A_1, \ldots, A_k) \in \Lambda_k$ for all $k \in \mathbb{N}$ and $y \in \bigcap_{k\in\mathbb{N}} A'_k$ and $(A'_1, \ldots, A'_k) \in \Lambda_k$ for all $k \in \mathbb{N}$. Since $x \in s_n(q_n) \cap s_n(A_1, \ldots, A_n)$ and $y \in s_n(q_n) \cap s_n(A'_1, \ldots, A'_n)$, it follows from the second part of Property (a_n), that $(A_1, \ldots, A_n) = q_n = (A'_1, \ldots, A'_n)$. Hence, by the definition of d, $d(x,y) < 1/n$. Since $x, y \in C \cap s_n(q_n)$ were arbitrary, $\text{diam}[C \cap s_n(q_n)] \leq 1/n < \varepsilon$; which completes the proof. $\square$

The significance of Theorem 1.4.4 stems from the fact that it associates two disparate properties, namely, it relates the global property of fragmentability (of the whole space) to the local property of a winning strategy for the player Ω in the $\mathcal{G}(\tau, \tau')$-game. Indeed, Theorem 1.4.4 is usually employed as a tool for showing that certain topological spaces are fragmentable, by simply displaying a winning strategy for the player Ω in the $\mathcal{G}(\tau, \tau')$-game played on (X, τ, τ').

Exercise 1.4.5. *Suppose that (X, τ) is a regular Hausdorff topological space. Show that if (X, τ) has a countable network, then (X, τ) is fragmented by a metric whose topology on X is at least as strong as τ. Recall that $\mathcal{N} \subseteq 2^X$ is a network for the topology τ if for each $x \in X$ and $U \in \tau$ with $x \in U$, there exists an $N \in \mathcal{N}$ such that $x \in N \subseteq U$.*

Hint: Construct a winning strategy for the player Ω in the $\mathcal{G}(\tau, \tau)$-game played on (X, τ) and then apply Theorem 1.4.4.

In the ensuing lemma we shall exploit the following notion. Suppose that A is a nonempty subset of $\mathbb{R}^Y$, for some nonempty set Y and U, is a nonempty subset of Y. Then $\mathrm{Var}_A(U) := \sup\{|f(u) - f(u')| : u, u' \in U \text{ and } f \in A\}$. If, in addition, the set Y is endowed with a topology τ' then we write $\omega_A(x) := \inf\{\mathrm{Var}_A(U) : x \in U \in \tau'\}$, for each $x \in Y$.

Lemma 1.4.6. *Let (Y, τ') be a nonempty compact topological space and let $\varnothing \neq A \subseteq C(Y)$. If $0 < \varepsilon$ and W is an open neighbourhood of $\Delta_Y := \{(x, y) \in Y^2 : x = y\}$ then there exists a nonempty relatively $\tau_p(Y)$-open subset U of A and a point $(x_0, y_0) \in W$ such that*

(i) $\min\{1, s - \varepsilon\} < |g(x_0) - g(y_0)|$ *for all* $g \in U$ *and*

(ii) $\|\cdot\|_\infty - \mathrm{diam}[U] \leq 2s + \varepsilon$ *where,* $s := \sup\{\omega_A(x) : x \in Y\}$.

Proof. Choose $x \in Y$ such that $\min\{1, s - \varepsilon\} < \omega_A(x)$. Since $(x, x) \in W$, and W is open in the product topology there exists an open neighbourhood V of x such that $(x, x) \in V \times V \subseteq W$. Now,

$$\min\{1, s - \varepsilon\} < \omega_A(x) \leq \mathrm{Var}_A(V) = \sup\{|g(y) - g(y')| : y, y \in V \text{ and } g \in A\}.$$

Choose $f \in A$ and $x_0, y_0 \in V$ such that $\min\{1, s - \varepsilon\} < |f(x_0) - f(y_0)|$. Let

$$U^* := \{g \in A : \min\{1, s - \varepsilon\} < |g(x_0) - g(y_0)|\}.$$

Note that U^* is nonempty, as $f \in U^*$ and U^* is relatively $\tau_p(Y)$-open in A. Let $s^* := \sup\{\omega_{U^*}(x) : x \in Y\} \leq s$. For each $x \in X$, we may choose an open neighbourhood V_x of x such that $\mathrm{Var}_{U^*}(V_x) < (\omega_{U^*}(x) + \varepsilon/3) \leq (s^* + \varepsilon/3) \leq (s + \varepsilon/3)$. Let $\{V_{x_j} : 1 \leq j \leq n\}$ be a finite subcover of the open cover $\{V_x : x \in Y\}$ of Y. Let

$$U := \{g \in U^* : |g(x_j) - f(x_j)| < \varepsilon/6 \text{ for all } 1 \leq j \leq n\}.$$

Then $\varnothing \neq U$, as $f \in U$ and U is relatively $\tau_p(Y)$-open in U^* and hence, relatively $\tau_p(Y)$-open in A.

We claim that $\|\cdot\|_\infty - \operatorname{diam}[U] \leq 2s+\varepsilon$. To verify this claim, let us consider and $g, h \in U$ and any $x \in Y$. Then, since $\{V_{x_j} : 1 \leq j \leq n\}$ is a cover of Y, there exists a $1 \leq j_0 \leq n$ such that $x \in V_{x_{j_0}}$. Furthermore,

$$|g(x_{j_0}) - h(x_{j_0})| \leq |g(x_{j_0}) - f(x_{j_0})| + |f(x_{j_0}) - h(x_{j_0})| < \varepsilon/6 + \varepsilon/6 = \varepsilon/3$$

since $g, h \in U$. Now,

$$\begin{aligned} |g(x) - h(x)| &\leq |g(x) - g(x_{j_0})| + |g(x_{j_0}) - h(x_{j_0})| + |h(x_{j_0}) - h(x)| \\ &< (s + \varepsilon/3) + \varepsilon/3 + (s + \varepsilon/3) = 2s + \varepsilon. \end{aligned}$$

Since $x \in Y$ was arbitrary, $\|g-h\|_\infty < 2s+\varepsilon$ and since $g, h \in U$ were arbitrary too, $\|\cdot\|_\infty - \operatorname{diam}[U] \leq 2s+\varepsilon$. This completes the proof. □

The following important theorem relies heavily upon Theorem 1.4.4.

Theorem 1.4.7 ([32])**.** *Let (Y, τ') be a nonempty compact Hausdorff topological space and let X be a nonempty subset of $C(Y)$. Then the following are equivalent:*

(i) *X is fragmented by a metric whose topology on X is at least as strong as the $\|\cdot\|_\infty$-topology on X;*

(ii) *X is fragmented by a metric whose topology on X is at least as strong as the pointwise toplogy on X;*

(iii) *The player Ω possesses a strategy $\omega := (\omega_n : n \in \mathbb{N})$ for the $\mathcal{G}(\tau_p(Y), \tau_p(Y))$-game played on $(X, \tau_p(Y))$ such that, for every ω-play $(A_n : n \in \mathbb{N})$ either, (a) $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$ or (b) $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$ and every sequence $(x_n : n \in \mathbb{N})$ with $x_n \in A_n$ for all $n \in \mathbb{N}$, has a $\tau_p(K)$-cluster-point in $C(Y)$.*

Proof. Clearly $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ follows from Theorem 1.4.4. So we need only show that $(iii) \Rightarrow (i)$; which is what we do now. We shall apply Theorem 1.4.4. More precisely, we will construct a winning strategy $s := (s_n : n \in \mathbb{N})$ for the player Ω in the $\mathcal{G}(\tau_p(Y), \|\cdot\|_\infty)$-game played on X and then deduce from Theorem 1.4.4 that $(X, \tau_p(Y))$ is fragmented by a metric d whose topology on X is at least as strong as the supremum topology on X.

Let $\omega := (\omega_n : n \in \mathbb{N})$ be a strategy for the player Ω such that, for every ω-play $(A_n : n \in \mathbb{N})$ either $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$ or else, $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$ and every sequence $(x_n : n \in \mathbb{N})$ with $x_n \in A$ for all $n \in \mathbb{N}$, has a $\tau_p(Y)$-cluster-point in $C(Y)$.

Step 1. Let A_1 be a nonempty subset of X and let $A := \omega_1(A_1)$. Note that A is a nonempty relatively $\tau_p(Y)$-open subset of A_1.

Let $t_{(A_1)} := \sup\{\omega_A(x) : x \in Y\}$, $f_{(A_1)} \in A$, $\varepsilon := 1$ and

$$W := \{(x, y) \in Y^2 : |f_{(A_1)}(x) - f_{(A_1)}(y)| < 1\}.$$

Then $0 < \varepsilon$ and W is an open neighbourhood of Δ_Y. Hence, by Lemma 1.4.6, there exists a nonempty relatively $\tau_p(Y)$-open subset U of A and a point $(x_{(A_1)}, y_{(A_1)}) \in W$ such that

(i) $\min\{1, t_{(A_1)} - 1\} < |g(x_{(A_1)}) - g(y_{(A_1)})|$ for all $g \in U$ and

(ii) $\|\cdot\|_\infty - \text{diam}[U] \leq 2t_{(A_1)} + 1$.

Let $s_1(A_1) := U$. Note that this is well-defined since U is nonempty and a relatively $\tau_p(Y)$-open subset of $A = w_1(A_1)$ and hence, a relatively open $\tau_p(Y)$-open subset of A_1.

Now, let $n \in \mathbb{N}$ and suppose that s_j, $(x_{(A_1,\ldots,A_j)}, y_{(A_1,\ldots,A_j)}) \in Y^2$, $f_{(A_1,\ldots,A_j)} \in C(Y)$ and $t_{(A_1,\ldots,A_j)} \in [0, \infty]$ have been defined for every partial s-play $(A_1, \ldots, A_j)$ of length j with $1 \leq j \leq n$ so that:

(a_j) $(A_1, \ldots, A_j)$ is a partial ω-play and $s_j(A_1, \ldots, A_j) \subseteq \omega_j(A_1, \ldots, A_j)$;

(b_j) $t_{(A_1,\ldots,A_j)} := \sup\{\omega_A(x) : x \in Y\}$, where $A := \omega_j(A_1, \ldots, A_j)$;

(c_j) $f_{(A_1,\ldots,A_j)} \in A_j$;

(d_j) $|f_{(A_1,\ldots,A_k)}(x_{(A_1,\ldots,A_j)}) - f_{(A_1,\ldots,A_k)}(y_{(A_1,\ldots,A_j)})| < 1/j$ for all $1 \leq k \leq j$;

(e_j) $\min\{1, t_{(A_1,\ldots,A_j)} - 1/j\} < |g(x_{(A_1,\ldots,A_j)}) - g(y_{(A_1,\ldots,A_j)})|$ for all $g \in s_j(A_1, \ldots, A_j)$;

(f_j) $\text{diam}[s_j(A_1, \ldots, A_j)] \leq 2t_{(A_1,\ldots,A_j)} + 1/j$.

Step $n+1$. Suppose that $(A_1, \ldots, A_{n+1})$ is a partial s-play of length $n+1$. Then

$$\varnothing \neq A_{n+1} \subseteq s_n(A_1, \ldots, A_n).$$

By (a_n), $(A_1, \ldots, A_n)$ is a partial ω-play and $s_n(A_1, \ldots, A_n) \subseteq \omega_n(A_1, \ldots, A_n)$. Thus, $(A_1, \ldots, A_{n+1})$ is a partial ω-play (and the first part of (a_{n+1}) is satisfied). Therefore, $\varnothing \neq A := \omega_{n+1}(A_1, \ldots, A_{n+1})$ is well-defined. Note also that A is relatively $\tau_p(Y)$-open in A_{n+1}. Let $t_{(A_1,\ldots,A_{n+1})} := \sup\{\omega_A(x) : x \in Y\}$, $f_{(A_1,\ldots,A_{n+1})} \in A$, $\varepsilon := 1/(n+1)$ and

$$W := \{(x, y) \in Y^2 : |f_{(A_1,\ldots,A_k)}(x) - f_{(A_1,\ldots,A_k)}(y)| < 1/(n+1) \text{ for all } 1 \leq k \leq (n+1)\}.$$

Note that (b_{n+1}) and (c_{n+1}) are satisfied. Then $0 < \varepsilon$ and W is an open neighbourhood of Δ_Y. Hence, by Lemma 1.4.6, there exists a nonempty relatively $\tau_p(Y)$-open subset U of A and a point $(x_{(A_1,\ldots,A_{n+1})}, y_{(A_1,\ldots,A_{n+1})}) \in W$ such that

(i) $\min\{1, t_{(A_1,\ldots,A_{n+1})} - 1/(n+1)\} < |g(x_{(A_1,\ldots,A_{n+1})}) - g(y_{(A_1,\ldots,A_{n+1})})|$ for all $g \in U$ and

(ii) $\|\cdot\|_\infty - \text{diam}[U] \leq 2t_{(A_1,\ldots,A_{n+1})} + 1/(n+1)$.

Note that since $(x_{(A_1,\ldots,A_{n+1})}, y_{(A_1,\ldots,A_{n+1})}) \in W$, (d_{n+1}) is satisfied.

Let $s_{n+1}(A_1, \ldots, A_{n+1}) := U$. Note that this is well-defined since U is nonempty and a relatively $\tau_p(Y)$-open subset of $A = w_{n+1}(A_1, \ldots, A_{n+1})$ and hence, a relatively open $\tau_p(Y)$-open subset of A_{n+1}. Finally, let us note that: the second part of (a_{n+1}) is satisfied; (e_{n+1}) is satisfied by (i) above and (f_{n+1}) is satisfied by (ii) above.

This completes the definition of $s := (s_n : n \in \mathbb{N})$. We now show that s is a winning strategy for the player Ω in the $\mathcal{G}(\tau_p(Y), \|\cdot\|_\infty)$-game played on X. To this end, let

$(A_n : n \in \mathbb{N})$ be an arbitrary s-play. If $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$ then Ω wins this play. So, let us consider the case when $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$. At this point we shall also simplify our notation.

For each $n \in \mathbb{N}$, let $x_n := x_{(A_1,\ldots,A_n)}$, $y_n := y_{(A_1,\ldots,A_n)}$, $t_n := t_{(A_1,\ldots,A_n)}$ and $f_n := f_{(A_1,\ldots,A_n)}$. By construction

$$\min\{1, t_n - 1/n\} < |f_k(x_n) - f_k(y_n)| \quad \text{if } 1 \leq n < k \qquad (*)$$

since $f_k \in A_k \subseteq A_{n+1} \subseteq s_n(A_1, \ldots, A_n)$, see Property (e_n) and

$$|f_n(x_k) - f_n(y_k) < 1/k \quad \text{if } 1 \leq n \leq k \qquad (**)$$

see, Property (d_k). Since $f_n \in A_n$ for all $n \in \mathbb{N}$, $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$ and $(A_n : n \in \mathbb{N})$ is an ω-play, see Property (a_n), $(f_n : n \in \mathbb{N})$ had a $\tau_p(Y)$-cluster-point $f_\infty \in C(Y)$. Moreover, by inequality $(*)$

$$\min\{1, t_n - 1/n\} \leq |f_\infty(x_n) - f_\infty(y_n)| \quad \text{for all } n \in \mathbb{N}. \qquad (***)$$

On the other hand, since Y^2 is compact and $(x_n, y_n) \in Y^2$ for all $n \in \mathbb{N}$, the sequence $((x_n, y_n) : n \in \mathbb{N})$ has a cluster-point $(x_\infty, y_\infty) \in Y^2$ and furthermore, by $(**)$, we have that

$$0 \leq |f_n(x_\infty) - f_n(y_\infty)| \leq 0 \quad \text{for all } n \in \mathbb{N}.$$

That is, $f_n(x_\infty) = f_n(y_\infty)$ for all $n \in \mathbb{N}$. Therefore,

$$f_\infty(x_\infty) = f_\infty(y_\infty). \qquad (****)$$

We now claim that $\inf_{n\in\mathbb{N}} t_n = \lim_{n\to\infty} t_n = 0$. Firstly, note that $0 \leq t_{n+1} \leq t_n$ for all $n \in \mathbb{N}$. Therefore, by the Monotone Convergence Theorem, $\lim_{n\to\infty} t_n$ exists and equals $\inf_{n\in\mathbb{N}} t_n$. Next, suppose, in order to obtain a contradiction, that $0 < t := \inf_{n\in\mathbb{N}} t_n$. Then $\min\{1, t - 1/n\} \leq \min\{1, t_n - 1/n\}$ for all $n \in \mathbb{N}$. If $N \in \mathbb{N}$ and $2/t < N$ then

$$0 < \min\{1, t/2\} \leq \min\{1, t - 1/N\} \leq \min\{1, t - 1/n\} \quad \text{for all } N \leq n.$$

Hence, by $(***)$

$$0 < \min\{1, t/2\} \leq |f_\infty(x_n) - f_\infty(y_n)| \quad \text{for all } N < n.$$

Since, $(x, y) \mapsto |f_\infty(x) - f_\infty(y)|$, is continuous on Y^2, $0 < \min\{1, t/2\} \leq |f_\infty(x_\infty) - f_\infty(y_\infty)|$; which contradicts Equation $(****)$. Therefore t must equal zero, i.e., $\lim_{n\to\infty} t_n = 0$.

It now follows from Property (f_n) that

$$0 \leq \|\cdot\|_\infty - \text{diam}[A_{n+1}] \leq \|\cdot\|_\infty - \text{diam}[s_n(A_1, \ldots, A_n)] \leq 2t_n + 1/n \quad \text{for all } n \in \mathbb{N}.$$

Thus, $\lim_{n\to\infty} \|\cdot\|_\infty - \text{diam}[A_n] = 0$. This shows that the s-play $(A_n : n \in \mathbb{N})$ is won by the player Ω in this case too. This completes the proof. □

Corollary 1.4.8. *Suppose that (Y_1, τ_1') and (Y_2, τ_2') are nonempty compact Hausdorff topological spaces. Suppose also that X_1 is a nonempty subset of $C(Y_1)$ and X_2 is a nonempty subset of $C(Y_2)$. If $(X_1, \tau_p(Y_1))$ is homeomorphic to $(X_2, \tau_p(Y_2))$ then $(X_1, \tau_p(Y_1))$ is fragmented by a metric whose topology on X_1 is at least as strong as the $\|\cdot\|_\infty$ topology on X_1 if, and only if, $(X_2, \tau_p(Y_2))$ is fragmented by a metric whose topology on X_2 is at least as strong as the $\|\cdot\|_\infty$ topology on X_2.*

Proof. This follows directly from Theorem 1.4.7, since Property (ii) of Theorem 1.4.7 is preserved by $\tau_p(Y_1)$-to-$\tau_p(Y_2)$ homeomorphisms. $\square$

Corollary 1.4.9. *Suppose that (Y, τ') is a nonempty compact Hausdorff topological space. Then $(B_{C(Y)}, \tau_p(Y))$ is fragmented by a metric whose topology on $B_{C(Y)}$ is at least as strong as the $\|\cdot\|_\infty$ topology on $B_{C(Y)}$ if, and only if, $(C(Y), \tau_p(Y))$ is fragmented by a metric whose topology on $C(Y)$ is at least as strong as the $\|\cdot\|_\infty$ topology on $C(Y)$.*

Proof. Clearly, if $(C(Y), \tau_p(Y))$ is fragmented by a metric whose topology on $C(Y)$ is at least as strong as the $\|\cdot\|_\infty$ topology on $C(Y)$ then $(B_{C(Y)}, \tau_p(Y))$ is fragmented by a metric whose topology on $B_{C(Y)}$ is at least as strong as the $\|\cdot\|_\infty$ topology on $B_{C(Y)}$. So we consider the converse statement. Suppose that $(B_{C(Y)}, \tau_p(Y))$ is fragmented by a metric whose topology on $B_{C(Y)}$ is at least as strong as the $\|\cdot\|_\infty$ topology on $B_{C(Y)}$. Then $B := \{f \in C(Y) : \|f\|_\infty < 1\} \subseteq B_{C(Y)}$ will also be fragmented by a metric whose topology on B is at least as strong as the $\|\cdot\|_\infty$ topology on B. We now claim that $(B, \tau_p(Y))$ is homeomorphic to $(C(Y), \tau_p(Y))$. To see this, let us first consider a homeomorphism $g : (-1, 1) \to (-\infty, \infty)$ [e.g. $g : (-1, 1) \to (-\infty, \infty)$, defined by, $g(x) := \tan(\frac{\pi}{2}x)$ for all $x \in (-1, 1)$]. Then define $T_g : B \to C(Y)$ by,

$$T_g(f)(x) := (g \circ f)(x) = g(f(x)) \quad \text{for all } x \in Y.$$

Note that T_g is well-defined, i.e., $T_g(f) \in C(Y)$ for all $f \in B$, since $f(x) \in \mathrm{Dom}(g)$ for all $x \in Y$ and $(g \circ f)$ is continuous, as it is a composition of continuous functions. Furthermore, it is easy to see that T_g is 1-to-1 and onto. In fact $(T_g)^{-1} = T_{g^{-1}}$. In addition to this, it is not difficult to show that both $T_g : (B, \tau_p(Y)) \to (C(Y), \tau_p(Y))$ and $T_{g^{-1}} : (C(Y), \tau_p(Y)) \to (B, \tau_p(Y))$ are continuous. This completes the proof the claim.

The result now follows from Corollary 1.4.8. $\square$

As our next application of Theorem 1.4.7 we have the following result.

Theorem 1.4.10 ([32])**.** *Let (Y, τ') be a nonempty compact Hausdorff space. If $(B_{C(Y)}, \tau_p(Y))$ has a countable separation index (in some Hausdorff compactification) then $(C(Y), \tau_p(Y))$ is fragmented by a metric whose topology on $C(Y)$ is at least as strong as the $\|\cdot\|_\infty$-topology on $C(K)$.*

Proof. By Corollary 1.4.9, it is sufficient to show that $(B_{C(Y)}, \tau_p(Y))$ is fragmented by a metric whose topology on $B_{C(Y)}$ is at least as strong as the $\|\cdot\|_\infty$ topology on $B_{C(Y)}$. Furthermore, by Theorem 1.4.7, it is sufficient to show that the player Ω possesses a strategy $\omega := (\omega_n : n \in \mathbb{N})$ for the $\mathcal{G}(\tau_p(Y), \tau_p(Y))$-game played on $(B_{C(Y)}, \tau_p(Y))$ such that, for every ω-play $(A_n : n \in \mathbb{N})$ either, (i) $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$ or (ii) $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$ and every sequence $(x_n : n \in \mathbb{N})$ with $x_n \in A_n$ for all $n \in \mathbb{N}$, has a $\tau_p(K)$-cluster-point in $C(Y)$.

We now inductively define a strategy $\omega := (\omega_n : n \in \mathbb{N})$ that possesses the properties described above. Let $(O_n : n \in \mathbb{N})$ be a countable family of open subset of $(\beta(B_{C(Y)}), \tau_\beta)$ that separate $B_{C(Y)}$ from $\beta(B_{C(Y)}) \setminus B_{C(Y)}$.

Step 1. Suppose that A_1 is a nonempty subset of $B_{C(Y)}$. If $A_1 \cap O_1 = \varnothing$ then $\overline{A_1}^{\tau_\beta} \cap O_1 = \varnothing$. In this case, let $\omega_1(A_1) := A_1$. If $A_1 \cap O_1 \neq \varnothing$ then choose $x \in A_1 \cap O_1$. Then choose a τ_β-open neighbourhood U of x such that $x \in U \subseteq \overline{U}^{\tau_\beta} \subseteq O_1$. In this case, let $\omega_1(A_1) := A_1 \cap U$. Note that in both cases $\overline{\omega_1(A_1)}^{\tau_\beta}$ is not separated by the set O_1.

Now, let $n \in \mathbb{N}$ and suppose that ω_j has been defined for every partial ω-play of length j with $1 \leq j \leq n$ so that either, $\overline{\omega_j(A_1, \ldots, A_j)}^{\tau_\beta} \cap O_j$ or $\overline{\omega_j(A_1, \ldots, A_j)}^{\tau_\beta} \subseteq O_j$.

Step $n+1$. Let $(A_1, \ldots, A_{n+1})$ be a partial ω-play of length $n+1$. If $A_{n+1} \cap O_{n+1} = \varnothing$ then $\overline{A_{n+1}}^{\tau_\beta} \cap O_{n+1} = \varnothing$. In this case, let $\omega_{n+1}(A_1, \ldots, A_{n+1}) := A_{n+1}$. If $A_{n+1} \cap O_{n+1} \neq \varnothing$ then choose $x \in A_{n+1} \cap O_{n+1}$. Then choose a τ_β-open neighbourhood U of x such that $x \in U \subseteq \overline{U}^{\tau_\beta} \subseteq O_{n+1}$. In this case, let $\omega_{n+1}(A_1, \ldots, A_{n+1}) := A_{n+1} \cap U$. Note that in both cases, $\overline{\omega_{n+1}(A_1, \ldots, A_{n+1})}^{\tau_\beta}$ is not separated by the set O_{n+1}.

This completes the definition of the strategy $\omega := (\omega_n : n \in \mathbb{N})$. Let $(A_n : n \in \mathbb{N})$ be an ω-play and suppose that $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$. We need to show that every sequence $(x_n : n \in \mathbb{N})$ with $x_n \in A_n$ for all $n \in \mathbb{N}$, has a $\tau_p(K)$-cluster-point in $C(Y)$. So, let $(x_n : n \in \mathbb{N})$ be a sequence with $x_n \in A_n$ for all $n \in \mathbb{N}$. Since $(\beta(B_{C(Y)}), \tau_\beta)$ is compact the sequence $(x_n : n \in \mathbb{N})$ has a cluster-point $x_\infty \in \beta(B_{C(Y)})$. We need to show that $x_\infty \in B_{C(Y)}$. Fix $n \in \mathbb{N}$ and let $n < k$. Then

$$x_k \in A_k \subseteq A_n \subseteq \overline{A_n}^{\tau_\beta}.$$

Therefore, $x_\infty \in \overline{A_n}^{\tau_\beta}$. Since $n \in \mathbb{N}$ was arbitrary, $x_\infty \in \bigcap_{n\in\mathbb{N}} \overline{A_n}^{\tau_\beta}$. Since the points of $C := \bigcap_{n\in\mathbb{N}} \overline{A_n}^{\tau_\beta}$ are not distinguished by the sets $(O_n : n \in \mathbb{N})$ either, $C \subseteq B_{C(Y)}$ or $C \subseteq \beta(B_{C(Y)}) \setminus B_{C(Y)}$. However, as $\varnothing \neq \bigcap_{n\in\mathbb{N}} A_n \subseteq C \cap B_{C(Y)}$, $C \subseteq B_{C(Y)}$. In particular, $x_\infty \in B_{C(Y)}$. Therefore, x_∞ is a $\tau_p(Y)$-cluster-point of the sequence $(x_n : n \in \mathbb{N})$. □

A related notion to fragmentability, that was first considered in [25], (even though the papers [22, 23] came out earlier, due to publication delays), is that of σ-fragmentability.

A topological space (X, τ) is said to be *sigma-fragmented* by a metric d defined on X if, for every $0 < \varepsilon$ there exists a countable family $\{X_n^\varepsilon : n \in \mathbb{N}\}$ of subsets of X such that (i) $X = \bigcup_{n\in\mathbb{N}} X_n^\varepsilon$ (i.e., $\{X_n^\varepsilon : n \in \mathbb{N}\}$ is a cover of X) and (ii) for every $n \in \mathbb{N}$, and every nonempty subset A of X_n^ε, there exists an open subset U such that $A \cap U \neq \varnothing$ and $d - \text{diam}(A \cap U) < \varepsilon$.

Sometimes we write σ-fragmented in place of "sigma-fragmented". Sigma-fragmentability was studied extensively in the 1990's, particularly in the setting of Banach spaces and $C(Y)$-spaces. See for example, [16, 22–28, 30, 32, 37]. During this period the connection between renorming theory and sigma-fragmentability was studied, as well as, the connection between the co-Namioka property and sigma-fragmentiability, [21, 36, 38]. For a paper that contains a little of the history of fragmentability/sigma-fragmentability, in its introduction, see [34].

In the last part of this section, we will explore the connection between fragmentability and σ-fragmentability.

Proposition 1.4.11 ([32]). *Let (X, τ) be a topological space that is fragmented by a metric d, whose topology on X, is at least as strong as the topology generated by some other (but not necessarily distinct) metric ρ defined on X. Then (X, τ) is sigma-fragmented by ρ.*

Proof. For every $x \in X$ and $r \in (0, \infty)$ define, $B_d(x, r) := \{y \in X : d(y, x) < r\}$. Given $0 < \varepsilon$, put $X_n^\varepsilon := \{x \in X : \rho - \text{diam}[B_d(x, 1/n)] < \varepsilon\}$. Since the d-topology on X is at least as strong as the ρ-topology on X, $X = \bigcup_{n\in\mathbb{N}} X_n^\varepsilon$. Fix $n \in \mathbb{N}$ and let A be a nonempty subset of X_n^ε. Since (X,τ) is fragmented by d there exists a nonempty relatively τ-open subset U of A with $d-\text{diam}[U] < 1/n$. Choose $x_0 \in U \subseteq X_n^\varepsilon$. Then $U \subseteq B_d(x_0, 1/n)$. By the definition of the set X_n^ε

$$\rho - \text{diam}[U] \leq \rho - \text{diam}[B_d(x_0, 1/n)] < \varepsilon.$$

That is, (X,τ) is sigma-fragmented by the metric ρ. □

Of particular interest to us is the following corollary.

Corollary 1.4.12. *Let (Y,τ') be a nonempty compact Hausdorff space. If $(C(Y),\tau_p(Y))$ is fragmented by a metric d, whose topology on $C(Y)$ is at least as strong as the $\|\cdot\|_\infty$-topology on $C(K)$, then $(C(Y),\tau_p(Y))$ is sigma-fragmented by the metric generated by $\|\cdot\|_\infty$.*

Proof. We define $\rho : C(Y)\times C(Y) \to [0,\infty)$ by, $\rho(f,g) := \|f-g\|_\infty$ for all $f,g \in C(Y)$. Then ρ is a metric on $C(Y)$. We now just apply Proposition 1.4.11. □

Proposition 1.4.13 ([32])**.** *Let (X,τ,τ') be a bitopological space with the property that:*

(i) *(X,τ') is T_1 and*

(ii) *for every $U \in \tau'$ and every $x \in U$ there exists a $V \in \tau'$ such that $x \in V \subseteq \overline{V}^\tau \subseteq U$.*

If (X,τ) is sigma-fragmented by a metric d, whose topology on X, is at least as strong as τ', then (X,τ) is fragmented by a metric whose topology on X, is at least as strong as τ'.

Proof. We shall appeal to Theorem 1.4.4. Indeed, to prove the theorem, we will construct a winning strategy $s := (s_n : n \in \mathbb{N})$ for the player Ω in the $\mathcal{G}(\tau,\tau')$-game played on (X,τ,τ'), but first we shall make some preliminary observations and remarks. Let $\{(m_k,n_k) \in \mathbb{N}\times\mathbb{N} : k \in \mathbb{N}\}$ be an enumeration of $\mathbb{N}\times\mathbb{N}$ and for each $n \in \mathbb{N}$, let $\{X_{(m,n)} \in 2^X : m \in \mathbb{N}\}$ be a cover of X such that, for every $m \in \mathbb{N}$ and every nonempty subset A of $X_{(m,n)}$ there exists a nonempty relatively τ-open subset U of A such that $d-\text{diam}[U] < 1/n$.

Step 1. Let A_1 be a nonempty subset of X. We shall consider two cases:

(i) if $A_1 \not\subseteq \overline{A_1 \cap X_{(m_1,n_1)}}^\tau$ then we define $s_1(A_1) := A_1 \setminus \overline{A_1 \cap X_{(m_1,n_1)}}^\tau$. Note that $s_1(A_1)$ is a nonempty relatively τ-open subset of A_1;

(ii) if $A_1 \subseteq \overline{A_1 \cap X_{(m_1,n_1)}}^\tau$ then there exists a τ-open subset U of X such that $U \cap (A_1 \cap X_{(m_1,n_1)}) \neq \varnothing$ and $d-\text{diam}[U \cap (A_1 \cap X_{(m_1,n_1)})] < 1/n_1$. In this case we define $s_1(A_1) := U \cap A_1$. Note that $s_1(A_1)$ is a nonempty relatively τ-open subset of A_1.

So either $s_1(A_1) \cap X_{(m_1,n_1)} = \varnothing$, or else, $s_1(A_1) \subseteq \overline{s_1(A_1) \cap X_{(m_1,n_1)}}^\tau$.

Now, let $k \in \mathbb{N}$ and suppose that s_j has been defined for every partial s-play $(A,\ldots,A_j)$ of length j with $1 \leq j \leq k$ so that: either

(a_j) $s_j(A_1,\ldots,A_j) \cap X_{(m_j,n_j)} = \varnothing$, or else,

(b_j) $d-\text{diam}[s_j(A_1,\ldots,A_j) \cap X_{(m_j,n_j)}] < \frac{1}{n_j}$ and $s_j(A_1,\ldots,A_j) \subseteq \overline{s_j(A_1,\ldots,A_j) \cap X_{(m_j,n_j)}}^\tau$.

Step $k+1$. Let $(A_1, \ldots, A_{k+1})$ be a partial s-play of length $k+1$. We shall consider two cases.

(i) if $A_{k+1} \not\subseteq \overline{A_{k+1} \cap X_{(m_{k+1}, n_{k+1})}}^{\tau}$ then we define

$$s_{k+1}(A_1, \ldots, A_{k+1}) := A_{k+1} \setminus \overline{A_{k+1} \cap X_{(m_{k+1}, n_{k+1})}}^{\tau}.$$

Note that $s_{k+1}(A_1, \ldots, A_{k+1})$ is a nonempty relatively τ-open subset of A_{k+1};

(ii) if $A_{k+1} \subseteq \overline{A_{k+1} \cap X_{(m_{k+1}, n_{k+1})}}^{\tau}$ then there exists a τ-open subset U of X such that $U \cap (A_{k+1} \cap X_{(m_{k+1}, n_{k+1})}) \neq \varnothing$ and $d - \text{diam}[U \cap (A_{k+1} \cap X_{(m_{k+1}, n_{k+1})})] < 1/n_{k+1}$. In this case we define

$$s_{k+1}(A_1, \ldots, A_{k+1}) := U \cap A_{k+1}.$$

Note that $s_{k+1}(A_1, \ldots, A_{k+1})$ is a nonempty relatively τ-open subset of A_{k+1}.

Finally, observe that either (a_{k+1}) or (b_{k+1}) is satisfied.

This completes the definition of $s := (s_n : n \in \mathbb{N})$. So it remains to show that s is indeed a winning strategy for the player Ω in the $\mathcal{G}(\tau, \tau')$-game played in (X, τ, τ'). To this end, let $(A_n : n \in \mathbb{N})$ be an arbitrary s-play in the $\mathcal{G}(\tau, \tau')$-game played in (X, τ, τ'). If $\bigcap_{n \in \mathbb{N}} A_n = \varnothing$ then Ω wins this play. So let us consider the case when $\bigcap_{n \in \mathbb{N}} A_n \neq \varnothing$.

Let $x \in \bigcap_{n \in \mathbb{N}} A_n$. To verify that the s-play $(A_n : n \in \mathbb{N})$ is won by Ω it is sufficient to show, because of the assumed properties of τ and τ', that for each τ'-open neighbourhood V of x there exists an $n \in \mathbb{N}$ such that $A_n \subseteq \overline{V}^{\tau}$. [Take a minute to convince yourself that this is really true]. So let V an an arbitrary τ'-open neighbourhood of x. Since the d-topology on X is at least as strong as the τ'-topology on X there exists an $n' \in \mathbb{N}$ such that $B_d(x, 1/n') \subseteq V$. Now, since $\{X_{(m,n')} : m \in \mathbb{N}\}$ is a cover of X there exists a $m' \in \mathbb{N}$ such that $x \in X_{(m',n')}$.

Let $k \in \mathbb{N}$ be chosen so that $(m_k, n_k) = (m', n')$. Then $x \in s_k(A_1, \ldots, A_k) \cap X_{(m_k, n_k)}$. Therefore, Property (a_k) does not hold and so Property (b_k) must hold. That is,

$$d - \text{diam}[s_k(A_1, \ldots, A_k) \cap X_{(m_k, n_k)}] < 1/n_k = 1/n'$$

and

$$s_k(A_1, \ldots, A_k) \subseteq \overline{s_k(A_1, \ldots, A_k) \cap X_{(m_k, n_k)}}^{\tau}.$$

Therefore,

$$A_{k+1} \subseteq s_k(A_1, \ldots, A_k) \subseteq \overline{B_d(x, 1/n')}^{\tau} \subseteq \overline{V}^{\tau}$$

since $s_k(A_1, \ldots, A_k) \cap X_{(m_k, n_k)} \subseteq B_d(x, \frac{1}{n'})$. This completes the proof. □

Corollary 1.4.14. *Let (Y, τ') be a nonempty compact Hausdorff topological space. If $(C(Y), \tau_p(Y))$ is sigma-fragmented by the metric generated by $\|\cdot\|_\infty$, then $(C(Y), \tau_p(Y))$ is fragmented by a metric, whose topology on $C(Y)$, is at least as strong as the supremum norm topology on $C(Y)$.*

Proof. Let $\tau := \tau_p(Y)$, let d be the metric generated by $\|\cdot\|_\infty$ and let τ' be the topology generated d. Then we may simply apply Proposition 1.4.13. □

Corollary 1.4.15 ([22, Corollary 3.1]). *Let (Y, τ') be a nonempty compact Hausdorff topological space. If $(C(Y), \tau_p(Y))$ is sigma-fragmented by the metric generated by $\|\cdot\|_\infty$, then Y is a co-Namioka space.*

Proof. This follows from Corollary 1.4.14 and Corollary 1.4.2. □

Corollary 1.4.16. *Let (Y, τ') be a nonempty compact Hausdorff topological space. Then $(C(Y), \tau_p(Y))$ is σ-fragmented by the metric generated by $\|\cdot\|_\infty$ if, and only if, $(C(Y), \tau_p(Y))$ is fragmented by a metric, whose topology on $C(Y)$, is at least as strong as the supremum norm topology on $C(Y)$.*

Proof. Simply combine Corollary 1.4.12 and Corollary 1.4.14. □

1.5 Game characterisation of class($\mathcal{T}^*$)-spaces

In this section we present a characterisation of those compact Hausdorff spaces (Y, τ') such that $N(X, Y)$ holds for all weakly α-favourable spaces (X, τ).

Proposition 1.5.1. *Let $f : X \to Y$ be a quasicontinuous, open mapping from a topological space (X, τ) onto a topological space (Y, τ'). If (X, τ) is weakly α-favourable, then so is (Y, τ').*

Proof. Let $s := (s_n : n \in \mathbb{N})$ be a winning strategy for the player α in the $Ch(X)$-game played on (X, τ). We shall use s to inductively define a winning strategy $\sigma := (\sigma_n : n \in \mathbb{N})$ for the player α in the $Ch(Y)$-game played on (Y, τ').

Step 1. Let (B_1) be a partial σ-play of length 1 in the $Ch(Y)$-game played on (Y, τ'), i.e., B_1 is a nonempty open subset of Y. Since f is τ'-quasicontinuous and surjective there exists a nonempty τ-open subset $A_{(B_1)}$ of X such that $f(A_{(B_1)}) \subseteq B_1$. Let $\sigma_1(B_1) := f(s_1(A_{(B_1)}))$.

This definition is well-defined because (i) $f(s_1(A_{(B_1)}))$ is a nonempty open subset of Y, as f is an open mapping and $s_1(A_{(B_1})$ is a nonempty open subset of (X, τ) and (ii) $\sigma_1(B_1) \subseteq B_1$ as $\sigma_1(B_1) = f(s_1(A_{(B_1)})) \subseteq f(A_{(B_1)}) \subseteq B_1$.

Now, let $n \in \mathbb{N}$ and suppose that σ_j and $A_{(B_1,\ldots,B_j)}$ have been defined for each partial σ-play $(B_1, \ldots, B_j)$ of length j with $1 \le j \le n$ so that:

(i) $(A_{(B_1)}, \ldots, A_{(B_1,\ldots,B_j)})$ is a partial s-play;

(ii) $f(A_{(B_1,\ldots,B_j)}) \subseteq B_j$;

(iii) $\sigma_j(B_1, \ldots, B_j) := f(s_j(A_{(B_1)}, \ldots, A_{(B_1,\ldots,B_j)}))$.

Step $n+1$. Let $(B_1, \ldots, B_{n+1})$ be a partial σ-play of length $n+1$. Then B_{n+1} is a nonempty open subset of $\sigma_n(B_1, \ldots, B_n) = f(s_n(A_{(B_1)}, \ldots, A_{(B_1,\ldots,B_n)}))$. Therefore, since f is τ'-quasicontinuous, there exists a nonempty open subset $A_{(B_1,\ldots,B_{n+1})}$ of $s_n(A_{(B_1)}, \ldots, A_{(B_1,\ldots,B_n)})$ such that $f(A_{(B_1,\ldots,B_{n+1})}) \subseteq B_{n+1}$. Then define

$$\sigma_{n+1}(B_1, \ldots, B_{n+1}) := f(s_{n+1}(A_{(B_1)}, \ldots, A_{(B_1,\ldots,B_{n+1})})).$$

This definition is well-defined because (i) $\sigma_{n+1}(B_1, \ldots, B_{n+1})$ is a nonempty open subset of Y, as f is an open mapping and $s_{n+1}(A_{(B_1)}, \ldots, A_{(B_1,\ldots,B_{n+1})})$ is a nonempty open subset of (X, τ) and (ii) $\sigma_{n+1}(B_1, \ldots, B_{n+1}) \subseteq B'_{n+1}$ as

$$\sigma_{n+1}(B_1, \ldots, B_{n+1}) = f(s_{n+1}(A_{(B_1)}, \ldots, A_{(B_1,\ldots,B_{n+1})})) \subseteq f(A_{(B_1,\ldots,B_{n+1})}) \subseteq B_{n+1}.$$

This completes the definition of $\sigma := (\sigma_n : n \in \mathbb{N})$. So it remains to show that σ is indeed a winning strategy for the player α in the $Ch(Y)$-game played on (Y, τ'). Let $(B_n : n \in \mathbb{N})$ be an arbitrary σ-play. Then, by construction, $(A_{(B_1,\ldots,B_n)} : n \in \mathbb{N})$ is an s-play. Hence,

$$\varnothing \neq \bigcap_{n\in\mathbb{N}} A_{(B_1,\ldots,B_n)} = \bigcap_{n\in\mathbb{N}} s_n(A_{(B_1)}, \ldots, A_{(B_1,\ldots,B_n)}).$$

Furthermore,

$$\begin{aligned} \varnothing \neq f(\textstyle\bigcap_{n\in\mathbb{N}} s_n(A_{(B_1)}, \ldots, A_{(B_1,\ldots,B_n)})) &\subseteq \textstyle\bigcap_{n\in\mathbb{N}} f(s_n(A_{(B_1)}, \ldots, A_{(B_1,\ldots,B_n)})) \\ &= \textstyle\bigcap_{n\in\mathbb{N}} \sigma_n(B_1, \ldots, B_n) = \bigcap_{n\in\mathbb{N}} B_n; \end{aligned}$$

which shows that α wins this play. This completes the proof. □

Proposition 1.5.2. *Let $f : X \to Y$ be a quasicontinuous mapping from a topological space (X, τ) into a topological space (Y, τ'). If (X, τ) is weakly α-favourable, then so is the graph of f, $\mathrm{Gr}(f)$, endowed with the relative product topology of (X, τ) and (Y, τ').*

Proof. Let $f : X \to Y$ be a quasicontinuous mapping from a weakly α-favourable topological space (X, τ) into a topological space (Y, τ'). Let $g : X \to \mathrm{Gr}(f)$ be defined by, $g(x) := (x, f(x))$ for all $x \in X$. Then clearly, (i) g is surjective and (ii) g is open, since for any τ-open subset U of X, $g(U) = (U \times Y) \cap \mathrm{Gr}(f)$; which is obviously open in the relative product topology on $\mathrm{Gr}(f)$. We now show that g is quasicontinuous on (X, τ). To this end, let $x_0 \in X$, W be an open neighbourhood of $g(x_0)$ and U be an open neighbourhood of x_0. By the definition of the product topology, there exist open subsets U' of X and W' of Y such that

$$(x_0, f(x_0)) = g(x_0) \in U' \times W' \subseteq W.$$

Therefore, $x_0 \in U'$ and $f(x_0) \in W'$. Now, since $x_0 \in U \cap U'$ and $f(x_0) \in W'$, it follows from the quasicontinuous of f that there exists a nonempty open subset V of $U \cap U'$ such that $f(V) \subseteq W'$. Thus,

$$g(V) \subseteq (V \times W') \cap \mathrm{Gr}(f) \subseteq V \times W' \subseteq U' \times W' = W.$$

This shows that g is quasicontinuous on (X, τ). The result now follows from Proposition 1.5.1. □

Proposition 1.5.3 ([31])**.** *Let (Y, τ') be a topological space and let d be some metric defined on it. Then the following conditions are equivalent:*

(i) *every τ'-continuous mapping $f : X \to Y$ from a weakly α-favourable space (X, τ) into (Y, τ') is d-continuous at the points of a dense subset of (X, τ);*

(ii) *every* τ'*-quasicontinuous mapping* $f : X \to Y$ *from a weakly* α*-favourable space* (X, τ) *into* (Y, τ')*, is* d*-continuous at the points of a dense subset of* (X, τ)*.*

Proof. It is clear that condition (ii) implies condition (i). So it suffices to show that $(i) \Rightarrow (ii)$. In this direction suppose that (i) holds. Let $f : X \to Y$ be a τ'-quasicontinuous function acting from a weakly α-favourable space (X, τ) into (Y, τ'). Fix $0 < \varepsilon$ and consider the set

$$O_\varepsilon := \bigcup\{U \in \tau : d - \operatorname{diam}[f(U)] \leq \varepsilon\}.$$

Since O_ε is a union of τ-open subsets, it is itself, a τ-open subset of X. We claim that O_ε is also dense in (X, τ). In order to verify this claim let us consider an arbitrary nonempty τ-open subset U_0 of X.

Let $Z := \operatorname{Gr}(f)$ and let τ'' denote the relative product topology of (X, τ) and (Y, τ') on Z. From Proposition 1.5.2, (Z, τ'') is a weakly α-favourable topological space. Let $\pi : (X, \tau) \times (Y, \tau') \to (Y, \tau')$ be defined by, $\pi(x, y) := y$ for all $(x, y) \in X \times Y$. Clearly, π is τ'-continuous on $X \times Y$. Let $g := \pi|_Z$ be the restriction of π to Z. Then g is τ'-continuous on (Z, τ'').

Therefore, by condition (i), there exists a τ''-dense subset D of Z such that g is d-continuous at each point of D. In particular, there exists a point

$$z \in D \cap [(U_0 \times Y) \cap Z] = D \cap [(U_0 \times Y) \cap \operatorname{Gr}(f)],$$

since $(U_0 \times Y) \cap \operatorname{Gr}(f)$ is τ''-open in Z and nonempty. Moreover, since $z \in \operatorname{Gr}(f) \cap (U_0 \times Y)$ there exists a point $x_0 \in U_0$ such that $z = (x_0, f(x_0))$.

Since g is d-continuous at z, there exist open subsets U' of X and W' of Y such that

$$(x_0, f(x_0)) = z \in (U' \times W') \cap Z = (U' \times W') \cap \operatorname{Gr}(f) \quad \text{i.e., } x_0 \in U' \text{ and } f(x_0) \in W'$$

and

$$d - \operatorname{diam}[g((U' \times W') \cap Z)] < \varepsilon.$$

Now, as $x_0 \in U_0 \cap U'$ and $f(x_0) \in W'$, we have, by the τ' quasicontinuity of f, that there exists a nonempty τ-open subset V of $U_0 \cap U'$ such that $f(V) \subseteq W'$. Therefore, if $x, y \in V$, then $(x, f(x)), (y, f(y)) \in (U' \times W') \cap \operatorname{Gr}(f)$ and so

$$d(f(x), f(y)) = d(g(x, f(x)), g(x, f(y))) < \varepsilon.$$

Hence, $\varnothing \neq V \subseteq O_\varepsilon \cap U_0$. This shows that O_ε is dense in (X, τ). It now only remains to observe that $\bigcap_{n \in \mathbb{N}} O_{1/n}$ is dense in (X, τ) (since weakly α-favourable spaces are Baire spaces, see remarks just after Theorem 1.1.5) and f is d-continuous at each point of $\bigcap_{n \in \mathbb{N}} O_{1/n}$. □

Corollary 1.5.4. *Let* (Y, τ') *be a nonempty compact Hausdorff topological space and let* X *be a nonempty subset of* $C(Y)$*. Then the following conditions are equivalent:*

(i) *every* $\tau_p(Y)$*-continuous mapping* $f : Z \to X$ *from a weakly* α*-favourable space* (Z, τ) *into* $C(Y)$ *is* $\|\cdot\|_\infty$*-continuous at the points of a dense subset of* (Z, τ)*;*

(ii) *every* $\tau_p(Y)$*-quasicontinuous mapping* $f : Z \to X$ *from a weakly* α*-favourable space* (Z, τ) *into* X*, is* $\|\cdot\|_\infty$*-continuous at the points of a dense subset of* (Z, τ)*.*

Next, we explore the connection between norm continuity and the $\mathcal{G}(\tau_p(Y), \|\cdot\|_\infty)$-game.

Lemma 1.5.5. *Let* (X, τ, τ') *be a bitopological space and let* $t := (t_n : n \in \mathbb{N})$ *be a strategy for the player* Σ *in the* $\mathcal{G}(\tau, \tau')$*-game played on* (X, τ, τ')*. Then* (P, d) *is a nonempty complete metric space, where* P *denotes the set of all* t*-plays and* $d : P \times P \to [0, \infty)$ *is defined by,*

$$d(p, p') := \begin{cases} 0 & \text{if } p = p' \\ 1/n & \text{if } p \neq p'. \end{cases}$$

where, $p := (B_k : k \in \mathbb{N})$*,* $p' := (B'_k : k \in \mathbb{N})$ *and* $n := \min\{i \in \mathbb{N} : B_i \neq B'_i\}$*.*

Furthermore, for every partial t*-play* $(B_1, \ldots, B_n)$ *there exists a* t*-play* $p := (B'_k : k \in \mathbb{N})$ *such that* $B_k = B'_k$ *for all* $1 \leq k \leq n$*, i.e., every partial* t*-play has at least one continuation to a full* t*-play.*

Proof. We will prove the last claim of the lemma and the proof that P is nonempty first. Let $n \in \mathbb{N}$ and let $(B_1, \ldots, B_n)$ be a partial t-play in the $\mathcal{G}(\tau, \tau')$-game played on (X, τ, τ'). Let $p := (B'_k : k \in \mathbb{N})$ be defined in two parts. Firstly, if $1 \leq k \leq n$, then $B'_k := B_k$. Note, of course, that $(B'_1, \ldots, B'_n)$ is a partial t-play.

If $n < k$ then we define B'_k inductively as follows. Let $n \leq k$ and suppose that $(B_1, \ldots, B'_k)$ is a partial t-play then $B'_{k+1} := t_{k+1}(B'_1, \ldots, B'_k)$ is well-defined and $(B'_1, \ldots, B'_{k+1})$ is a partial t-play. Indeed, this follows from the fact that $t_{k+1}(B'_1, \ldots, B'_k)$ is a nonempty relatively τ-open subset of $t_{k+1}(B'_1, \ldots, B'_k)$. Hence, $t_{k+1}(B'_1, \ldots, B'_k)$ is a valid move for the player Ω in the $\mathcal{G}(\tau, \tau')$-game. This defines the play $p := (B'_k : k \in \mathbb{N})$. It follows from the construction that p is a t-play.

Next we will show that (P, d) is a metric space.

(i) Clearly, $0 \leq d(p, p')$ for all $p, p' \in P$. Moreover, it follows directly from the definition of d that $d(p, p') = 0$, if and only if, $p = p'$.

(ii) The symmetry of d also follows directly from the definition of d. As is usually the case, the triangle inequality is the most difficult property to check.

(iii) Let $p, p', p'' \in P$. We will show that $d(p, p'') \leq d(p, p') + d(p', p'')$. Suppose that $p := (B_k : k \in \mathbb{N})$, $p' := (B'_k : k \in \mathbb{N})$ and $p'' := (B''_k : k \in \mathbb{N})$.

Let $n_{p,p''} := \min\{i \in \mathbb{N} : B_i \neq B''_i\}$, $n_{p,p'} := \min\{i \in \mathbb{N} : P_i \neq P'_i\}$ and $n_{p',p''} := \min\{i \in \mathbb{N} : P'_i \neq P''_i\}$.
If $i < \min\{n_{p,p'}, n_{p',p''z}\}$ then $B_i = B'_i$ and $B'_i = B''_i$. That is, $B_i = B''_i$. Therefore,

$$\{i \in \mathbb{N} : B_i \neq B''_i\} \subseteq \{i \in \mathbb{N} : \min\{n_{p,p'}, n_{p',p''}\} \leq i\}.$$

Hence, $\min\{n_{p,p'}, n_{p',p''}\} \leq n_{p,p''}$. Thus, either

$$d(p, p'') = 1/n_{p,p''} \leq 1/n_{p,p'} = d(p, p') \quad \text{or} \quad d(p, p'') = 1/n_{p,p''} \leq 1/n_{p',p''} = d(p', p'').$$

It now follows that $d(p, p'') \leq d(p, p') + d(p', p'')$.

Finally, we need to show that (P, d) is complete. To do this end, let $(p^n : n \in \mathbb{N})$ be a Cauchy sequence in (P, d). For each $n \in \mathbb{N}$, let $p^n := (B_k^n : k \in \mathbb{N})$. Since $(p^n : n \in \mathbb{N})$ be a Cauchy sequence there exists a strictly increasing sequence $(n_k : k \in \mathbb{N})$ of natural numbers such that $d - \text{diam}[\{p^i : n_k \leq i\}] < 1/k$ for all $k \in \mathbb{N}$. In particular,

$$d(p^{n_k}, p^i) < 1/k \quad \text{for all } k \in \mathbb{N} \text{ and all } n_k \leq i.$$

Therefore,

$$B_j^{n_k} = B_j^i \quad \text{for all } k \in \mathbb{N}, \text{ all } 1 \leq j \leq k \text{ and all } n_k \leq i.$$

By setting $j := k$ we get that

$$B_k^{n_k} = B_k^i \quad \text{for all } k \in \mathbb{N} \text{ and all } n_k \leq i. \qquad (*)$$

We now define a new sequence $p := (B_k : k \in \mathbb{N})$ by, $B_k := B_k^{n_k}$ for all $k \in \mathbb{N}$. Notice that for each fixed $k \in \mathbb{N}$, it follows from the definition of p and Equation $(*)$ (with k replaced by j and i replaced by n_k) that

$$B_j = B_j^{n_j} = B_j^{n_k} \quad \text{for all } 1 \leq j \leq k \qquad (**)$$

since $n_j \leq n_k$. Now, for each $k \in \mathbb{N}$,

$$B_k = B_k^{n_k} \subseteq t_k(B_1^{n_k}, \ldots, B_{k-1}^{n_k}) = t_k(B_1, \ldots, B_{k-1}).$$

Therefore, p is a t-play. Finally, note that $d(p^i, p) < 1/k$ for all $n_k \leq i$. To see this, fix $k \in \mathbb{N}$ and let $1 \leq j \leq k$. Then, by the definition of p and Equation $(*)$, (with k replaced by j) we have that

$$B_j = B_j^{n_j} = B_j^i \quad \text{for all } n_j \leq i.$$

Since $n_j \leq n_k$ it follows that $B_j = B_j^i$ for all $n_k \leq i$. This show that $(p^n : n \in \mathbb{N})$ converges to $p \in P$. □

In order to delve deeper into the connection between norm continuity of $\tau_p(Y)$-continuous mappings and the $\mathcal{G}(\tau_p(Y), \|\cdot\|_\infty)$-game we will need to recall some definitions from set-valued analysis.

We say shall say that a set-valued mapping $F : X \to 2^Y$ acting from a topological space (X, τ) into subsets of a topological space (Y, τ') is a *τ'-minimal mapping* if: for each pair of open subsets U of X and W of Y such that $F(U) \cap W \neq \varnothing$, there exists a nonempty open subset V of U such that $F(V) \subseteq W$.

The following three exercises establish some of the most fundamental properties of minimal mappings.

Exercise 1.5.6. *Let $F : X \to 2^T$ be a set-valued mapping acting from a topological space (X, τ) into subsets of a topological space (Y, τ'). Show that if D is a dense subset of (X, τ) and F is a τ'-minimal mapping, then $F|_D : D \to 2^Y$ defined by, $F|_D(x) := F(x)$ for all $x \in D$, is also a τ'-minimal mapping.*

Exercise 1.5.7. *Let $F : X \to 2^T$ be a τ'-minimal mapping acting from a topological space (X, τ) into subsets of a topological space (Y, τ'). Show that if $G : X \to 2^Y$ is a set-valued mapping and $G(x) \subseteq F(x)$ for all $x \in X$, i.e., $\mathrm{Gr}(G) \subseteq \mathrm{Gr}(F)$, then G is also a τ'-minimal mapping.*

Exercise 1.5.8. *Let $f : X \to Y$ be a function acting between topological spaces (X, τ) and (Y, τ') and let $F :\to 2^Y$ be defined by, $F(x) := \{f(x\}$ for all $x \in X$. The F is τ'-minimal if, and only if, f is τ'-quasicontinuous on (X, τ).*

Lemma 1.5.9. *Let (X, τ, τ') be a bitopological space and let $t := (t_n : n \in \mathbb{N})$ be a winning strategy for the player Σ in the $\mathcal{G}(\tau, \tau')$-game played on (X, τ, τ'). Let P denote the space of all t-plays endowed with the metric d, defined in Lemma 1.5.5. Then the mapping $F : P \to 2^X$ defined by, $F(p) := \bigcap_{n\in\mathbb{N}} B_n$, where $p := (B_n : n \in \mathbb{N})$ is a τ-minimal mapping with nonempty images. Furthermore, if $p \in P$, $n \in \mathbb{N}$ and $A \subseteq X$ with $F(B(p, 1/n)) \subseteq A$, then $B_{n+1} \subseteq \overline{A}^{\tau}$, where $p := (B_n : n \in \mathbb{N})$.*

Proof. Since t is a winning strategy for the player Σ in the $\mathcal{G}(\tau, \tau')$-game played on (X, τ, τ'), $\bigcap_{n\in\mathbb{N}} B_n \neq \varnothing$ for each t-play $(B_n : n \in \mathbb{N}) \in P$. Therefore, F has nonempty images.

Let U be an open subset of P and W be a τ-open subset of X such that $F(U) \cap W \neq \varnothing$. Choose $p := (B_n : n \in \mathbb{N}) \in U$ such that $F(p) \cap W = \bigcap_{n\in\mathbb{N}} B_n \cap W \neq \varnothing$, i.e., $B_n \cap W \neq \varnothing$ for each $n \in \mathbb{N}$. Since U is open there exists a $k \in \mathbb{N}$ such that $B(p, 1/k) \subseteq U$. Then

$$(B_1, \ldots, B_k, B_{k+1} \cap W)$$

is a partial t-play of length $k + 1$, since $B_{k+1} \cap W$ is a nonempty relatively open subset of $t_{k+1}(B_1, \ldots, B_k)$. Let $p' := (B'_n : n \in \mathbb{N})$ be a continuation of $(B_1, \ldots, B_k, B_{k+1} \cap W)$. Note that by Lemma 1.5.5 such a continuation exists. Then $p' \in B(p, 1/k)$ since $B_n = B'_n$ for all $1 \leq n \leq k$. Furthermore, if $V := B(p, 1/(k+1))$ then, V is a nonempty open subset of $B(p, 1/k)$ and $F(V) \subseteq W$. Indeed, if $p'' := (B''_n : n \in \mathbb{N}) \in V = B(p', 1/(k + 1))$ then $B''_n = B'_n$ for all $1 \leq n \leq k + 1$ and so

$$F(p'') = \textstyle\bigcap_{n\in\mathbb{N}} B''_n \subseteq B''_{k+1} = B'_{k+1} = (B_{k+1} \cap W) \subseteq W.$$

To justify the last assertion of this lemma consider $p := (B_k : k \in \mathbb{N}) \in P$, $n \in \mathbb{N}$ and $A \subseteq X$ such that $F(B(p, 1/n)) \subseteq A$ and suppose, in order to obtain a contradiction, that $B_{n+1} \not\subseteq \overline{A}^{\tau}$. Then $(B_1, \ldots, B_n, B_{n+1} \setminus \overline{A}^{\tau})$ is a partial t-play of length $(n+1)$, since $B_{n+1} \setminus \overline{A}^{\tau}$ is a nonempty relatively τ-open subset of $t_{n+1}(B_1, \ldots, B_n)$. Let $p' := (B'_n : n \in \mathbb{N})$ be any continuation of $(B_1, \ldots, B_n, B_{n+1} \setminus \overline{A}^{\tau})$. Note that by Lemma 1.5.5 such a continuation exists. Moreover, $p' \in B(p, 1/n)$ since $B_k = B'_k$ for all $1 \leq k \leq n$. However,

$$F(p') = \textstyle\bigcap_{k\in\mathbb{N}} B'_k \subseteq B'_{n+1} = (B_{n+1} \setminus \overline{A}^{\tau}) \subseteq (X \setminus \overline{A}^{\tau}) \subseteq (X \setminus A);$$

which contradicts the assumption that $\varnothing \neq F(p') \subseteq F(B(p, 1/n)) \subseteq A$. □

Suppose that (X, τ) and (Y, τ') are topological spaces and $\Phi : X \to 2^Y$. If $x_0 \in X$ then we say that Φ is *τ'-upper semicontinuous at x_0* if, for every $W \in \tau'$ that contains $\Phi(x_0)$ there exists a neighbourhood U of x_0 such that $\Phi(U) \subseteq W$. Here we are using the notation

$\Phi(U) := \bigcup\{\Phi(u) : u \in U\}$. If Φ is τ'-upper semicontinuous at every point of X then we say that Φ *is τ'-upper semicontinuous on* X. When there is no ambiguity concerning the topology τ', we shall simply say that Φ is upper semicontinuous on X.

From this definition one can immediately deduce the following.

Exercise 1.5.10. *Suppose that (X, τ) and (Y, τ') are topological spaces and $\Phi : X \to 2^Y$. Show that $\Phi : X \to 2^Y$ is upper semicontinuous on X if, and only if, for each open subset W of Y, $\{x \in X : \Phi(x) \subseteq W\}$ is an open subset of X.*

An important bridge between the study of set-valued mappings (and minimal mappings in particular) and quasicontinuous mappings, is the notion of a selection. A function $s : X \to Y$ acting from a set X into a set Y is called a *selection* of a set-valued mapping $\Phi : X \to 2^Y$ if, $s(x) \in \Phi(x)$ for all $x \in X$. In the case when (X, τ) and (Y, τ') are topological spaces and Φ is a τ'-minimal mapping, it is easy, but important, to see that every selection $s : X \to Y$ of Φ, is τ'-quasicontinuous on (X, τ). Of particular significance to us is the following proposition.

Proposition 1.5.11. *Let (Y, τ, τ') be a bitopological space with the property that:*

(a) *(Y, τ') is T_1 and*

(b) *for every $U \in \tau'$ and every $x \in U$ there exists a $V \in \tau'$ such that $x \in V \subseteq \overline{V}^{\tau} \subseteq U$.*

Consider a τ-minimal mapping $\Phi : X \to 2^Y$ acting from a topological space (X, τ'') into subset of Y and a selection $s : D \to Y$ of Φ defined on a dense subset D of (X, τ''). Then,

(i) *for any open subset W of (X, τ''), $\Phi(W) \subseteq \overline{s(W \cap D)}^{\tau}$;*

(ii) *if $s : D \to Y$ is τ'-continuous at $x_0 \in D$, then $\Phi(x_0) = \{s(x_0)\}$ and Φ is τ'-upper semicontinuous at x_0.*

Proof. (i) Suppose, in order to obtain a contradiction, that W is an open subset of (X, τ'') and $\Phi(W) \not\subseteq \overline{s(W \cap D)}^{\tau}$. Then, since Φ is a τ-minimal mapping there exists a nonempty open subset V of W such that $\Phi(V) \cap \overline{s(W \cap D)}^{\tau}$. Therefore,

$$\varnothing \neq s(V \cap D) \subseteq \Phi(V) \subseteq (Y \setminus \overline{s(W \cap D)}^{\tau}) \subseteq Y \setminus s(V \cap D);$$

which is impossible. Therefore, $\Phi(W) \subseteq \overline{s(U \cap D)}^{\tau}$.

(ii) Let $x_0 \in D$ and suppose that $s : D \to Y$ is τ'-continuous at x_0. It will be sufficient to show, because of the assumed properties of τ and τ', that for each τ'-open neighbourhood V of $s(x_0)$ there exists an open neighbourhood W of x_0 such that $\Phi(W) \subseteq \overline{V}^{\tau}$. In fact, because of part (i) it will be sufficient to that for each τ'-open neighbourhood V of $s(x_0)$ there exists an open neighbourhood W of x_0 such that $\sigma(W \cap D) \subseteq \overline{V}^{\tau}$. However, this follows directly from the fact that s is τ'-continuous at x_0. □

We can now state and prove the main theorem of this section.

Theorem 1.5.12 ([31]). *Let (Y, τ, τ') be a bitopological space with the property that:*

(a) (Y, τ') *is* T_1 *and*

(b) *for every* $U \in \tau'$ *and every* $x \in U$ *there exists a* $V \in \tau'$ *such that* $x \in V \subseteq \overline{V}^{\tau} \subseteq U$.

Then the following are equivalent:

(i) *the* $\mathcal{G}(\tau, \tau')$*-game played on* (Y, τ, τ') *is* Σ*-unfavourable;*

(ii) *every* τ*-quasicontinuous mapping* $f : X \to Y$ *from a complete metric space* (X, ρ) *into* Y *has at least one point of* τ'*-continuity;*

(iii) *every* τ*-quasicontinuous mapping* $f : X \to Y$ *from a weakly* α*-favourable space* (X, τ'') *into* Y *is* τ'*-continuous at each point of a dense subset of* (X, τ'').

Proof. *(i)* $\Rightarrow$ *(iii)*. Let $f : X \to Y$ be a τ-quasicontinuous mapping from a weakly α-favourable space (X, τ'') into Y and let W be a nonempty τ''-open subset of X. We wish to show that

$$\{x \in X : f \text{ is } \tau'\text{-continuous}\} \cap W \neq \varnothing.$$

To accomplish this, we will inductively define strategy $t := (t_n : n \in \mathbb{N})$ for the player Σ in the $\mathcal{G}(\tau, \tau')$-game played on (Y, τ, τ'), and then exploit the assumption that the $\mathcal{G}(\tau, \tau')$-game is Σ-unfavourable. Since (X, τ'') is weakly α-favourable there exists a winning strategy $s := (s_n : n \in \mathbb{N})$ for the player α in the $Ch(X)$-game played on (X, τ'').

Let $t_1(\emptyset) := f(W)$. Then $t_1(\emptyset)$ is a nonempty subset of Y and hence a valid move for the player Σ.

Step 1. Let (B_1) be a partial t-play of length 1 in the $\mathcal{G}(\tau, \tau')$-game played on (Y, τ, τ'), i.e., B_1 is a nonempty relatively τ-open subset of $t_1(\emptyset) = f(W)$. Since f is τ-quasicontinuous there exists a nonempty open subset $A_{(B_1)}$ of X such that $f(A_{(B_1)}) \subseteq B_1$. Let $t_2(B_1) := f(s_1(A_{(B_1)}))$. Then $t_2(B_1)$ is nonempty and $t_2(B_1) = f(s_1(A_{(B_1)})) \subseteq f(A_{(B_1)}) \subseteq B_1$.

Let $n \in \mathbb{N}$ and suppose that t_{j+1} and $A_{(B_1,\ldots,B_j)}$ have been defined for each partial t-play $(B_1, \ldots, B_j)$ of length j with $1 \leq j \leq n$ so that:

(i) $(A_{(B_1)}, \ldots, A_{(B_1,\ldots,B_j)})$ is a partial s-play;

(ii) $f(A_{(B_1,\ldots,B_j)}) \subseteq B_j$ and

(iii) $t_{j+1}(B_1, \ldots, B_j) := f(s_j(A_{(B_1)}, \ldots, A_{(B_1,\ldots,B_j)}))$.

Step $n + 1$. Let $(B_1, \ldots, B_{n+1})$ be a partial t-play of length $n + 1$. Then B_{n+1} is a nonempty relatively τ-open subset of $t_{n+1}(B_1, \ldots, B_n) = f(s_n(A_{(B_1)}, \ldots, A_{(B_1,\ldots,B_n)}))$. Therefore, since f is τ-quasicontinuous, there exists a nonempty open subset $A_{(B_1,\ldots,B_{n+1})}$ of $s_n(A_{(B_1)}, \ldots, A_{(B_1,\ldots,B_n)})$ such that $f(A_{(B_1,\ldots,B_{n+1})}) \subseteq B_{n+1}$. Then define,

$$t_{(n+1)+1}(B_1, \ldots, B_{n+1}) := f(s_{n+1}(A_{(B_1)}, \ldots, A_{(B_1,\ldots,B_{n+1})})).$$

This completes the definition of $t := (t_n : n \in \mathbb{N})$. Since the game $\mathcal{G}(\tau, \tau')$-game is Σ-unfavourable there exists a t-play $(B_n : n \in \mathbb{N})$ where the player Ω wins. However, by construction $(A_{(B_1,\ldots,B_n)} : n \in \mathbb{N})$ is an s-play. Hence,

$$\varnothing \neq \bigcap_{n \in \mathbb{N}} A_{(B_1,\ldots,B_n)} = \bigcap_{n \in \mathbb{N}} s_n(A_{(B_1)}, \ldots, A_{(B_1,\ldots,B_n)})$$

and so

$$\begin{aligned}\varnothing \neq f(\textstyle\bigcap_{n\in\mathbb{N}} s_n(A_{(B_1)},\dots,A_{(B_1,\dots,B_n)})) &\subseteq \textstyle\bigcap_{n\in\mathbb{N}} f(s_n(A_{(B_1)},\dots,A_{(B_1,\dots,B_n)}))\\ &= \textstyle\bigcap_{n\in\mathbb{N}} t_{n+1}(B_1,\dots,B_n) = \bigcap_{n\in\mathbb{N}} B_n.\end{aligned}$$

Therefore, $\bigcap_{n\in\mathbb{N}} B_n := \{y\}$ for some $y \in Y$ and for every $U \in \tau'$ with $y \in U$ there exists a $k \in \mathbb{N}$ such that $B_k \subseteq U$. Let $x \in \bigcap_{n\in\mathbb{N}} A_{(B_1,\dots,B_n)} \subseteq W$. Then $f(x) = y$ and for every $U \in \tau'$ with $f(x) \in U$ there exists a $k \in \mathbb{N}$ such that $f(A_{(B_1,\dots,B_k)}) \subseteq B_k \subseteq U$. This shows that f is τ'-continuous at $x \in W$; which in turn shows that $\{x \in X : f \text{ is } \tau'\text{-continuous}\}$ is dense in (X, τ'').

$(iii) \Rightarrow (ii)$ is obvious as every complete metric space is weakly α-favourable.

$(ii) \Rightarrow (i)$. Let $t := (t_n : n \in \mathbb{N})$ be an arbitrary strategy for the player Σ in the $\mathcal{G}(\tau, \tau')$-game played on (X, τ, τ'). For the sole purpose of obtaining a contradiction, let us assume that t is not a winning strategy for the player Σ. Let P denote the set of all t-plays.Then, by Lemma 1.5.5, (P, d) is a nonempty complete metric space, where $d : P \times P \to [0, \infty)$ is defined by,

$$d(p, p') := \begin{cases} 0 & \text{if } p = p' \\ 1/n & \text{if } p \neq p'. \end{cases}$$

where, $p := (B_k : k \in \mathbb{N})$, $p' := (B'_k : k \in \mathbb{N})$ and $n := \min\{i \in \mathbb{N} : B_i \neq B'_i\}$.

Let $F : P \to 2^Y$ be defined by, $F(p) := \bigcap_{n\in\mathbb{N}} B_n$, where $p := (B_n : n \in \mathbb{N})$. Then by Lemma 1.5.9, we have that F is a τ-minimal mapping with nonempty images. Next, let $s : P \to Y$ be any selection of F. From the discussion just prior to Proposition 1.5.11, or simply, by direct observation, we have that s is a τ'-quasicontinuous mapping. Hence, by assumption, there exists a t-play $p_0 := (B_n : n \in \mathbb{N}) \in P$, where s is τ'-continuous. By Proposition 1.5.11 part (ii), it follows that $\{s(p_0)\} = F(p_0) = \bigcap_{n\in\mathbb{N}} B_n$ and F is τ'-upper semicontinuous at p_0. We now claim that the play p_0 is won by Ω. To confirm this assertion let U be any τ'-open neighbourhood of $s(p_0)$. From the assumed properties of the topologies τ and τ' there exists a $V \in \tau'$ such that $s(p_0) \in V \subseteq \overline{V}^{\tau} \subseteq U$. Since, F is τ'-upper semicontinuous at p_0 there exists an $n \in \mathbb{N}$ such that $F(B(p_0, 1/n)) \subseteq V$. Then, by Lemma 1.5.9, $B_{n+1} \subseteq \overline{V}^{\tau} \subseteq U$. This shows that the play p_0 is won by Ω; which in turn completes the proof. □

Theorem 1.5.13 ([31]). *Let (Y, τ') be a nonempty compact Hausdorff topological space and let X be a nonempty subset of $C(Y)$. Then the following conditions are equivalent:*

(i) *the player Σ possesses a winning strategy in the $\mathcal{G}(\tau_p(Y), \tau_p(Y))$-game played on $(X, \tau_p(Y), \tau_p(Y))$;*

(ii) *the player Σ possesses a winning strategy in the $\mathcal{G}(\tau_p(Y), \|\cdot\|_\infty)$-game played on $(X, \tau_p(Y), \|\cdot\|_\infty)$;*

(iii) *the player Σ possesses a strategy $\sigma := (\sigma_n : n \in \mathbb{N})$ for the $\mathcal{G}(\tau_p(Y), \tau_p(Y))$-game played on $(X, \tau_p(Y), \tau_p(Y))$ such that, for every σ-play $(B_n : n \in \mathbb{N})$; (a) $\bigcap_{n\in\mathbb{N}} B_n \neq \varnothing$ and (b) there exists a sequence $(f_n : n \in \mathbb{N})$ in $C(Y)$, with $f_n \in B_n$ for all $n \in \mathbb{N}$, that has no $\tau_p(Y)$-cluster points.*

Proof. The implications $(iii) \Rightarrow (i)$ and $(i) \Rightarrow (ii)$ are evident. Therefore, we need only prove that $(ii) \Rightarrow (iii)$.

To this end, suppose that (ii) holds. Let $t := (t_n : n \in \mathbb{N})$ be a winning strategy for the player Σ in the $\mathcal{G}(\tau_p(Y), \|\cdot\|_\infty)$-game played on $(X, \tau_p(Y), \|\cdot\|_\infty))$. We shall inductively define a strategy $\sigma := (\sigma_n : n \in \mathbb{N})$ for the player Σ in the $\mathcal{G}(\tau_p(Y), \tau_p(Y))$-game played on $(X, \tau_p(Y), \tau_p(Y))$.

Base Step. Let $\sigma_1(\emptyset) := t_1(\emptyset)$; which is a nonempty subset of X.

Step 1. Let (B_1) be an σ-play of length 1, i.e., B_1 is a nonempty relatively $\tau_p(Y)$-open subset of $\sigma_1(\emptyset)$. Let $A := t_2(B_1)$. Then A is well-defined, since B_1 is a nonempty relatively $\tau_p(Y)$-open subset of $t_1(\emptyset) = \sigma_1(\emptyset)$. Furthermore, A is a nonempty subset.
Let $s_{(B_1)} := \sup\{\omega_A(x) : x \in Y\}$, $f_{(B_1)} \in A$, $\varepsilon := 1$ and

$$W := \{(x, y) \in Y^2 : |f_{(B_1)}(x) - f_{(B_1)}(y)| < 1\}.$$

Then $0 < \varepsilon$ and W is an open neighbourhood of Δ_Y. Hence, by Lemma 1.4.6, there exists a nonempty relatively $\tau_p(Y)$-open subset U of A and a point $(x_{(B_1)}, y_{(B_1)}) \in W$ such that

(i) $\min\{1, s_{(B_1)} - 1\} < |g(x_{(B_1)}) - g(y_{(B_1)})|$ for all $g \in U$ and

(ii) $\|\cdot\|_\infty - \text{diam}[U] \leq 2s_{(B_1)} + 1$.

Let $\sigma_2(B_1) := U$. Note that $\sigma_2(B_1)$ is a nonempty relatively $\tau_p(Y)$-open subset of $t_2(B_1)$.

Now, let $n \in \mathbb{N}$ and suppose that σ_{j+1}, $(x_{(B_1,\ldots,B_j)}, y_{(B_1,\ldots,B_j)}) \in Y^2$, $f_{(B_1,\ldots,B_j)} \in C(Y)$ and $s_{(B_1,\ldots,B_j)} \in [0, \infty]$ have been defined for every partial σ-play $(B_1, \ldots, B_j)$ of length j with $1 \leq j \leq n$ so that:

(a_j) $(B_1, \ldots, B_j)$ is a partial t-play and $\sigma_{j+1}(B_1, \ldots, B_j)$ is a nonempty relatively $\tau_p(Y)$-open subset of $t_{j+1}(B_1, \ldots, B_j)$;

(b_j) $s_{(B_1,\ldots,B_j)} := \sup\{\omega_A(x) : x \in Y\}$, where $A := t_{j+1}(B_1, \ldots, B_j)$;

(c_j) $f_{(B_1,\ldots,B_j)} \in B_j$;

(d_j) $|f_{(B_1,\ldots,B_k)}(x_{(B_1,\ldots,B_j)}) - f_{(B_1,\ldots,B_k)}(y_{(B_1,\ldots,B_j)})| < 1/j$ for all $1 \leq k \leq j$;

(e_j) $\min\{1, s_{(B_1,\ldots,B_j)} - 1/j\} < |g(x_{(B_1,\ldots,B_j)}) - g(y_{(B_1,\ldots,B_j)})|$ for all $g \in \sigma_{j+1}(B_1, \ldots, B_j)$;

(f_j) $\text{diam}[\sigma_{j+1}(B_1, \ldots, B_j)] \leq 2s_{(B_1,\ldots,B_j)} + 1/j$.

Step $n + 1$. Suppose that $(B_1, \ldots, B_{n+1})$ is a partial σ-play of length $n + 1$. Then B_{n+1} is a nonempty relatively $\tau_p(Y)$-open subset of $\sigma_{n+1}(B_1, \ldots, B_n)$, which in turn, is by (a_n), a relatively $\tau_p(Y)$-open subset of $t_{n+1}(B_1, \ldots, B_n)$. Therefore, B_{n+1} is a nonempty relatively $\tau_p(Y)$-open subset of $t_{n+1}(B_1, \ldots, B_n)$. Thus, $(B_1, \ldots, B_{n+1})$ is a partial t-play, since by (a_n), $(B_1, \ldots, B_n)$ is a partial t-play. This shows that the first part of (a_{n+1}) is satisfied.

Let $A := t_{(n+1)+1}(B_1, \ldots, B_{n+1})$. Note that A is nonempty subset of B_{n+1}.
Let $s_{(B_1,\ldots,B_{n+1})} := \sup\{\omega_A(x) : x \in Y\}$, $f_{(B_1,\ldots,A_{B+1})} \in A$, $\varepsilon := 1/(n + 1)$ and

$$W := \{(x, y) \in Y^2 : |f_{(B_1,\ldots,B_k)}(x) - f_{(B_1,\ldots,B_k)}(y)| < 1/(n + 1) \text{ for all } 1 \leq k \leq (n + 1)\}.$$

Note that (b_{n+1}) and (c_{n+1}) are satisfied. Then $0 < \varepsilon$ and W is an open neighbourhood of Δ_Y. Hence, by Lemma 1.4.6, there exists a nonempty relatively $\tau_p(Y)$-open subset U of A and a point $(x_{(B_1,\ldots,B_{n+1})}, y_{(B_1,\ldots,B_{n+1})}) \in W$ such that

(i) $\min\{1, s_{(B_1,\ldots,B_{n+1})} - 1/(n+1)\} < |g(x_{(B_1,\ldots,B_{n+1})}) - g(y_{(B_1,\ldots,B_{n+1})})|$ for all $g \in U$ and

(ii) $\|\cdot\|_\infty - \operatorname{diam}[U] \leq 2t_{(B_1,\ldots,B_{n+1})} + 1/(n+1)$.

Note that since $(x_{(B_1,\ldots,B_{n+1})}, y_{(B_1,\ldots,B_{n+1})}) \in W$, (d_{n+1}) is satisfied.

Let $\sigma_{(n+1)+1}(B_1, \ldots, B_{n+1}) := U$. Note that U is a relatively $\tau_p(Y)$-open subset of $A = t_{(n+1)+1}(B_1, \ldots, B_{n+1})$ and hence, the second part of (a_{n+1}) is satisfied. Note also that (e_{n+1}) is satisfied by (i) above and (f_{n+1}) is satisfied by (ii) above.

This completes the definition of $\sigma := (\sigma_n : n \in \mathbb{N})$.

We now show that σ has the desired properties. Let $(B_n : n \in \mathbb{N})$ be an arbitrary σ-play. By construction, $(B_n : n \in \mathbb{N})$ is also a t-play. Therefore, $\bigcap_{n\in\mathbb{N}} B_n \neq \varnothing$, since t is a winning strategy for the player Σ. Furthermore, $f_{(B_1,\ldots,B_n)} \in B_n$ for all $n \in \mathbb{N}$, however, the sequence $(f_{(B_1,\ldots,B_n)} : n \in \mathbb{N})$ does not possess any $\tau_p(Y)$-cluster points. To prove this last assertion, we will assume, for the purpose of obtaining a contradiction, that the sequence $(f_{(B_1,\ldots,B_n)} : n \in \mathbb{N})$ does in fact possess a $\tau_p(Y)$-cluster point, which we will call f_∞.

At this point we shall also simplify our notation.

For each $n \in \mathbb{N}$, let $x_n := x_{(B_1,\ldots,B_n)}$, $y_n := y_{(B_1,\ldots,B_n)}$, $s_n := s_{(B_1,\ldots,B_n)}$ and $f_n := f_{(B_1,\ldots,B_n)}$. By construction

$$\min\{1, s_n - 1/n\} < |f_k(x_n) - f_k(y_n)| \quad \text{if } 1 \leq n < k \qquad (*)$$

since $f_k \in B_k \subseteq B_{n+1} \subseteq \sigma_{n+1}(B_1, \ldots, B_n)$, see Property (e_n) and

$$|f_n(x_k) - f_n(y_k) < 1/k \quad \text{if } 1 \leq n \leq k \qquad (**)$$

see, Property (d_k). By inequality $(*)$

$$\min\{1, s_n - 1/n\} \leq |f_\infty(x_n) - f_\infty(y_n)| \quad \text{for all } n \in \mathbb{N}. \qquad (***)$$

On the other hand, since Y^2 is compact and $(x_n, y_n) \in Y^2$ for all $n \in \mathbb{N}$, the sequence $((x_n, y_n) : n \in \mathbb{N})$ has a cluster-point $(x_\infty, y_\infty) \in Y^2$ and furthermore, by $(**)$, we have that

$$0 \leq |f_n(x_\infty) - f_n(y_\infty)| \leq 0 \quad \text{for all } n \in \mathbb{N}.$$

That is, $f_n(x_\infty) = f_n(y_\infty)$ for all $n \in \mathbb{N}$. Therefore,

$$f_\infty(x_\infty) = f_\infty(y_\infty). \qquad (****)$$

We now claim that $\inf_{n\in\mathbb{N}} s_n = \lim_{n\to\infty} s_n = 0$. Firstly, note that $0 \leq s_{n+1} \leq s_n$ for all $n \in \mathbb{N}$. Therefore, by the Monotone Convergence Theorem, $\lim_{n\to\infty} s_n$ exists and equals $\inf_{n\in\mathbb{N}} s_n$. Next, suppose, in order to obtain a contradiction, that $0 < s := \inf_{n\in\mathbb{N}} s_n$. Then $\min\{1, s - 1/n\} \leq \min\{1, s_n - 1/n\}$ for all $n \in \mathbb{N}$. If $N \in \mathbb{N}$ and $2/s < N$ then

$$0 < \min\{1, s/2\} \leq \min\{1, s - 1/N\} \leq \min\{1, s - 1/n\} \quad \text{for all } N \leq n.$$

Hence, by $(***)$

$$0 < \min\{1, s/2\} \leq |f_\infty(x_n) - f_\infty(y_n)| \quad \text{for all } N < n.$$

Since, $(x, y) \mapsto |f_\infty(x) - f_\infty(y)|$, is continuous on Y^2, $0 < \min\{1, t/2\} \leq |f_\infty(x_\infty) - f_\infty(y_\infty)|$; which contradicts Equation $(****)$. Therefore s must equal zero, i.e., $\lim_{n\to\infty} s_n = 0$.

It now follows from Property (f_n) that

$$0 \leq \|\cdot\|_\infty - \text{diam}[B_{n+1}] \leq \|\cdot\|_\infty - \text{diam}[\sigma_{n+1}(B_1, \ldots, B_n)] \leq 2s_n + 1/n \quad \text{for all } n \in \mathbb{N}.$$

Thus, $\lim_{n\to\infty} \|\cdot\|_\infty - \text{diam}[B_n] = 0$. This shows that the t-play $(B_n : n \in \mathbb{N})$ is won by the player Ω; which contradicts the assumption that t is a winning strategy for the player Σ. Hence, our supposition that $(f_{(B_1,\ldots,B_n)} : n \in \mathbb{N})$ has a $\tau_p(Y)$-cluster point must have been false. □

We now state and prove our main theorem for this section of the monograph.

Theorem 1.5.14 ([31]). *Let (Y, τ') be a nonempty compact Hausdorff topological space and let X be a nonempty subset of $C(Y)$. Then the following conditions are equivalent:*

(i) *every $\tau_p(Y)$-quasicontinuous mapping $f : Z \to X$ from a complete metric space (Z, ρ) into X has at least one point of $\tau_p(Y)$-continuity;*

(ii) *the player Σ does not possess a winning strategy in the $\mathcal{G}(\tau_p(Y), \tau_p(Y))$-game played on $(X, \tau_p(Y), \tau_p(Y))$;*

(iii) *the player Σ does not possess a winning strategy in the $\mathcal{G}(\tau_p(Y), \|\cdot\|_\infty)$-game played on $(X, \tau_p(Y), \|\cdot\|_\infty)$;*

(iv) *every $\tau_p(Y)$-quasicontinuous mapping $f : Z \to X$ from a complete metric space (Z, ρ) into X has at least one point of $\|\cdot\|_\infty$-continuity;*

(v) *every $\tau_p(Y)$-quasicontinuous mapping $f : Z \to X$ from a weakly α-favourable space (Z, τ) into X is $\|\cdot\|_\infty$-continuous at each point of a dense subset of (Z, τ);*

(vi) *every $\tau_p(Y)$-continuous mapping $f : Z \to X$ from a weakly α-favourable space (Z, τ) into X is $\|\cdot\|_\infty$-continuous at each point of a dense and G_δ subset of (Z, τ).*

Proof. $(i) \Rightarrow (ii)$. This follows directly from Theorem 1.5.12.

$(ii) \Leftrightarrow (iii)$. This follows directly from Theorem 1.5.13.

$(iii) \Rightarrow (iv)$ and $(iv) \Rightarrow (v)$ follow directly from Theorem 1.5.12.

Now, $(v) \Leftrightarrow (vi)$ follows from Corollary 1.5.4 and the fact that the set of points of norm continuity always form a G_δ set.

$(v) \Rightarrow (i)$ is obvious, after one recalls that all complete metric spaces are weakly α-favourable. □

Motivated by Theorem 1.5.14 we introduce the following definition.

We shall say that a nonempty compact Hausdorff topological space (Y, τ') belongs to $\mathcal{T}^*$ if every $\tau_p(Y)$-continuous mapping $f : Z \to C(Y)$ from a weakly α-favourable space (Z, τ) into $C(Y)$ is $\|\cdot\|_\infty$-continuous at each point of a dense and G_δ subset of (Z, τ).

Clearly, $\mathcal{N}^* \subseteq \mathcal{T}^*$ and an example of R. Haydon, [20, Theorem 3.3], based upon a tree of Todorčević, [51] shows that $\mathcal{N}^*$ is, in fact, a proper subclass of $\mathcal{T}^*$.

Note also that any compact space $(Y, \tau') \in \mathcal{T}^* \setminus \mathcal{N}^*$ is an example of a space where the $\mathcal{G}(\tau_p(Y), \|\cdot\|_\infty)$-game played on $(C(Y), \tau_p(Y), \|\cdot\|_\infty)$ is undetermined, i.e., neither player, Σ nor Ω, possesses a winning strategy. Indeed, if the player Ω has a winning strategy in the $\mathcal{G}(\tau_p(Y), \|\cdot\|_\infty)$-game, then $C(Y)$ would be fragmented by a metric whose topology on $C(Y)$ would be at least as strong as the norm topology on $C(Y)$ (see Theorem 1.4.4), and so, (Y, τ') would be a co-Namioka space (see Corollary 1.4.2), which it is not. On the other hand, by Theorem 1.5.14, the player Σ does not possess winning strategy in the $\mathcal{G}(\tau_p(Y), \|\cdot\|_\infty)$-game either, since $(Y, \tau') \in \mathcal{T}^*$.

Thus, topological games, appear to be a natural tool for exploring the connection between separate and joint continuity.

Corollary 1.5.15 ([31])**.** *Suppose that (Y_1, τ_1') and (Y_2, τ_2') are nonempty compact Hausdorff topological spaces. Suppose also that X_1 is a nonempty subset of $C(Y_1)$ and X_2 is a nonempty subset of $C(Y_2)$. If $(X_1, \tau_p(Y_1))$ is homeomorphic to $(X_2, \tau_p(Y_2))$ then the following properties are equivalent:*

(i) *every $\tau_p(Y)$-continuous mapping $f : Z \to X_1$ from a weakly α-favourable space (Z, τ) into X_1 is $\|\cdot\|_\infty$-continuous at each point of a dense and G_δ subset of (Z, τ);*

(ii) *every $\tau_p(Y)$-continuous mapping $f : Z \to X_2$ from a weakly α-favourable space (Z, τ) into X_2 is $\|\cdot\|_\infty$-continuous at each point of a dense and G_δ subset of (Z, τ).*

In particular, if $C_p(Y_1)$ is homeomorphic to $C_p(Y_2)$, then $(Y_1, \tau_1') \in \mathcal{T}^*$ if, and only if, $(Y_2, \tau_2') \in \mathcal{T}^*$.

Proof. This holds because either of the conditions (i) or (ii) of Theorem 1.5.14 characterise the class $\mathcal{T}^*$ solely in terms of the $\tau_p(Y)$-topology. □

The fourth-coming

Lemma 1.5.16 ([7, Corollary C])**.** *Let (Y, τ) be a nonempty compact Hausdorff space and let D be a countable dense subset of Y. Then every subset of $C(Y)$ which is compact with respect to the $\tau_p(D)$-topology and Lindelöf with respect to the $\tau_p(Y)$-topology, is separable in $(C(K), \|\cdot\|_\infty)$.*

For the ensuing theorem we will let T denote the set of all finite sequences of 0's and 1's, including the sequence of length 0, which will be denoted by $\emptyset$. It $t \in T$, (i.e., $t := (t_1, t_2, \ldots, t_n)$ for some $n \in \mathbb{N}$, or $t = \emptyset$) then we define the *length* of t , denoted $|t|$, to be n if, $t := (t_1, t_2, \ldots, t_n)$ for some $n \in \mathbb{N}$ or 0 if $t = \emptyset$. We will also write $t1$ for the sequence $(t_1, t_2, \ldots, t_n, 1)$ and $t0$ for the sequence $(t_1, t_2, \ldots, t_n, 0)$. Of course, if $|t| = 0$ then $\emptyset 1 = (1)$ and $\emptyset 0 = (0)$.

Theorem 1.5.17 ([31,33])**.** *Let (Y, τ') be a nonempty compact Hausdorff space and let X be a Lindelöf subspace of $C_p(Y)$. Then every $\tau_p(Y)$-continuous mapping $f : Z \to X$ from a weakly α-favourable space (Z, τ) into X is $\|\cdot\|_\infty$-continuous at each point of a dense and G_δ subset of (Z, τ).*

Proof. We shall apply Theorem 1.5.14. Specifically, we shall use the fact that condition (iv) of Theorem 1.5.14 implies condition (vi) of Theorem 1.5.14. Let (Z, ρ) be a complete metric space and let $f : Z \to X$ be a $\tau_p(Y)$-quasicontinuous mapping. Fix $0 < \varepsilon$ and consider the set

$$O_\varepsilon := \bigcup\{U \in 2^Z : \ U \text{ is open and } \|\cdot\|_\infty - \text{diam}[f(U)] \leq \varepsilon\}.$$

Since O_ε is a union of open sets, it is itself, an open subset of Z. We claim O_ε is also dense in (Z, ρ). To verify this claim, we will assume, for the sole purpose of obtaining a contradiction, that O_ε is not dense in (Z, ρ). Then there exists a nonempty open subset W of Z such that $O_\varepsilon \cap W = \varnothing$. In particular, $\varepsilon < \|\cdot\|_\infty - \text{diam}[f(U)]$ for every nonempty open subset U of W.

We shall inductively (on the length $|t|$ of $t \in T$) define the following: nonempty open subsets U_t of Z, elements $x_t \in Y$ and dense open subsets $\{O_k^t : k \in \mathbb{N}\}$ of Z that fulfil the following properties:

(a) the mapping, $z \mapsto f(z)(x_t)$, is continuous at each point of $\bigcap_{k\in\mathbb{N}} O_k^t$;

(b) $\overline{U_{t0}} \cup \overline{U_{t1}} \subseteq U_t \cap \bigcap\{O_k^{t'} : 1 \leq k \leq |t|+1 \text{ and } |t'| \leq |t|\}$ and $\text{diam}[U_t] < 1/2^{|t|}$;

(c) if $h \in f(U_{t0})$ and $g \in f(U_{t1})$ then $\varepsilon \leq (g-h)(x_t)$.

First, let $U_\emptyset$ be any nonempty open subset of W with $\text{diam}[U_t] < 1 = 1/2^0$.

Base Step. By assumption $\varepsilon < \|\cdot\|_\infty - \text{diam}[f(U_\emptyset)]$, hence there exist $x, y \in U_\emptyset$ such that $\varepsilon < \|f(x) - f(y)\|_\infty$. After possibly interchanging the names of x and y we may choose $x_\emptyset \in Y$ such that $(f(x) - f(y))(x_\emptyset) = \varepsilon + \delta$ for some $0 < \delta$. Furthermore, since the mapping, $z \mapsto f(z)(x_\emptyset)$, is quasicontinuous, there exist dense open subsets $\{O_k^\emptyset : k \in \mathbb{N}\}$ of Z such that, $z \mapsto f(z)(x_\emptyset)$, is continuous at each point of $\bigcap_{k\in\mathbb{N}} O_k^\emptyset$. Let

$$U_{(1)} := \{z \in U_\emptyset : f(z)(x_\emptyset) > f(x)(x_\emptyset) - \frac{\delta}{2}\} \text{ and } U_{(0)} := \{z \in U_\emptyset : f(z)(x_\emptyset) < f(y)(x_\emptyset) + \frac{\delta}{2}\}.$$

Then, $\varepsilon \leq (g-h)(x_\emptyset)$ for all $h \in f(U_{(0)})$ and $g \in f(U_{(1)})$. Note that by possibly making $U_{(0)}$ and $U_{(1)}$ smaller we may assume that:

(i) $\overline{U_{(0)}} \cup \overline{U_{(1)}} \subseteq U_\emptyset \cap O_1^\emptyset$ and

(ii) $\text{diam}[U_{(0)}] < 1/2$ and $\text{diam}[U_{(1)}] < 1/2$.

Clearly, $U_{(0)}$ and $U_{(1)}$ are nonempty open subsets of $U_\emptyset$, $x_\emptyset \in Y$ and $\{O_k^\emptyset : k \in \mathbb{N}\}$ are dense open subset of Z, which together satisfy the properties (a), (b) and (c).

Now suppose that U_t, $x_{t'}$ and $\{O_k^{t'} : k \in \mathbb{N}\}$ have been defined for all $t \in T$ with $|t| \leq n$ and all $t' \in T$ with $|t'| < n$.

Inductive Step. Consider $t \in T$ of length n. By assumption $\varepsilon < \|\cdot\|_\infty - \text{diam}[f(U_t)]$, hence there exist $x, y \in U_t$ such that $\varepsilon < \|f(x) - f(y)\|_\infty$. After possibly interchanging the names of x and y we may choose $x_t \in Y$ such that $(f(x) - f(y))(x_t) = \varepsilon + \delta$ for some $0 < \delta$. Furthermore, since the mapping, $z \mapsto f(z)(x_t)$, is quasicontinuous, there exist dense open subsets $\{O_k^t : k \in \mathbb{N}\}$ of Z such that, $z \mapsto f(z)(x_t)$, is continuous at each point of $\bigcap_{k\in\mathbb{N}} O_k^t$. Let

$$U_{t1} := \{z \in U_t : f(z)(x_t) > f(x)(x_t) - \frac{\delta}{2}\} \text{ and } U_{t0} := \{z \in U_t : f(z)(x_t) < f(y)(x_t) + \frac{\delta}{2}\}.$$

Then, $\varepsilon \leq (g - h)(x_t)$ for all $h \in f(U_{t0})$ and $g \in f(U_{t1})$. Note that by possibly making U_{t0} and U_{t1} smaller we may assume that:

(i) $\overline{U_{t0}} \cup \overline{U_{t1}} \subseteq U_t \cap \bigcap\{O_k^{t'} : 1 \leq k \leq |t| + 1 \text{ and } |t'| \leq |t|\}$ and

(ii) $\text{diam}[U_{t0}] < 1/2^{|t0|}$ and $\text{diam}[U_{t1}] < 1/2^{|t1|}$.

Clearly, U_{t0} and U_{t1} are nonempty open subsets of U_t, $x_t \in Y$ and $\{O_k^t : k \in \mathbb{N}\}$ are dense open subset of Z, which together satisfy the properties (a), (b) and (c).

This completes the induction. Let $D := \{x_t : t \in T\}$ and let $K := \overline{D}$. By construction, the set $C := \bigcap_{n\in\mathbb{N}} C_n$, where $C_n := \bigcup_{|t|=n} \overline{U_t}$, is a closed and totally bounded subset of Z (and hence compact, since (Z, ρ) is complete). Furthermore, the construction also yields that for each $t \in T$, $C \subseteq \bigcap_{k\in\mathbb{N}} O_k^t$. Indeed, if $t \in T$ and and $k \in \mathbb{N}$, then $K \subseteq K_{m+1} \subseteq O_k^t$, where $m := \max\{k, |t|\}$. Thus, for each $t \in T$, the mapping, $z \mapsto f(z)(x_t)$, is continuous on C. Note also that for each pair of distinct points x and x' in C there exists a $t \in T$ such that $\varepsilon \leq |f(x)(x_t) - f(x')(x_t)|$. Next, we consider the mapping $\mathscr{R} : C(Y) \to C(K)$ defined by, $\mathscr{R}(f) := f|_K$. Then $(\mathscr{R} \circ f)(C)$ is a non-separable subset of $(C(K), \|\cdot\|_\infty)$ that is compact with respect to $\tau_p(D)$. Moreover, since that $\tau_p(D)$-topology on $C(K)$ is Hausdorff, $(\mathscr{R} \circ f)(C)$ is closed in the $\tau_p(D)$-topology and hence closed in the $\tau_p(K)$-topology on $C(K)$. However, by Lemma 1.5.16, this is impossible since $(\mathscr{R} \circ f)(C) \subseteq \mathscr{R}(L)$; which is Lindelöf. This shows that O_ε is dense in (Z, ρ). It now only remains to observe that f is $\|\cdot\|_\infty$-continuous at each point of $\bigcap_{n\in\mathbb{N}} O_{1/n}$. □

Corollary 1.5.18. *If (Y, τ') is a nonempty compact Hausdorff space and $C_p(Y)$ is Lindelöf, then $(Y, \tau') \in \mathcal{T}^*$.*

Proof. This follows directly from Theorem 1.5.17. □

Corollary 1.5.19 ([33])**.** *If (Z, τ) is weakly α-favourable and Lindelöf, then $(Z, \tau) \in \mathcal{N}$, i.e., (Z, τ) is a Namioka space.*

Proof. Let $f : Z \to C(Y)$ be a $\tau_p(Y)$-continuous mapping. Let $X := f(Z)$. Then X is a Lindelöf subset of $C_p(K)$. The result now follows form Theorem 1.5.17. □

We now demonstrate how Theorem 1.5.14 may be used, to show that certain compact spaces are not members of $\mathcal{T}^*$. Our results depend upon the following two lemmas.

Lemma 1.5.20 ([4])**.** *Let $x^* \in \ell^\infty(\mathbb{N})^*$ and let M be an infinite subset of $\mathbb{N}$. Then there exists an infinite subset $M' \subseteq M$ such that $|x^*(x)| < 1$ whenever $\|x\|_\infty \leq 2$ and $\text{supp}(x) \subseteq M'$. Here, $\text{supp}(x) := \{n \in \mathbb{N} : x_n \neq 0\}$.*

Proof. Suppose that the lemma is false. Take some $d \in \mathbb{N}$ with $2\|x^*\| < d$ and find a disjoint family $\{M_i : 1 \leq i \leq d\}$ of infinite subsets of M. Since each M_i, $i \in \{1, \ldots, d\}$ fails the property stated in the lemma, there are some elements $x_i \in \ell^\infty(\mathbb{N})$, $i \in \{1, \ldots, d\}$ such that (i) $\|x_i\|_\infty \leq 2$; (ii) $\text{supp}(x_i) \subseteq M_i$ and (iii) $1 \leq x^*(x_i)$. Then, for $x := \sum_{i=1}^d x_i$ we have that $\|x\|_\infty \leq 2$ and

$$2\|x^*\| < d \leq \textstyle\sum_{i=1}^d x^*(x_i) = x^*(x) \leq \|x^*\|\|x\|_\infty \leq 2\|x^*\|.$$

This contradiction completes the proof of the lemma. □

For the next lemma we need to introduce some more notion. Firstly, we shall denote by $P_\infty(\mathbb{N})$ the set of all subsets of $\mathbb{N}$ whose complement, in $\mathbb{N}$, is infinite. For each $M \in P_\infty(\mathbb{N})$ and each $x := (x_n : n \in \mathbb{N}) \in \ell^\infty(\mathbb{N})$ we denote by

$$S(x, M) := \{y \in \ell^\infty(\mathbb{N}) : \|y\|_\infty \leq 1 \text{ and } y_m = x_m \text{ for all } m \in M\}.$$

Note that for each $x \in \ell^\infty(\mathbb{N})$ and $M \in P_\infty(\mathbb{N})$, $\|\cdot\|_\infty - \text{diam}[S(x, M)] = 2$.

Lemma 1.5.21 ([32])**.** *Let $x \in \ell^\infty(\mathbb{N})$ and let $M \in P_\infty(\mathbb{N})$. Then for each weak open subset U of $\ell^\infty(\mathbb{N})$ with $U \cap S(x, M) \neq \varnothing$ there exists a $x' \in S(x, M)$ and a set $M' \in P_\infty(\mathbb{N})$, containing M, such that $S(x', M') \subseteq U \cap S(x, M)$.*

Proof. Let $x \in \ell^\infty(\mathbb{N})$, $M \in P_\infty(\mathbb{N})$ and let U be a weak open subset of $\ell^\infty(\mathbb{N})$ such that $U \cap S(x, M) \neq \varnothing$. Let x' be any element of $U \cap S(x, M)$. By the definition of the weak topology on $\ell^\infty(\mathbb{N})$ there exists x_i^*, $1 \leq i \leq k$ such that

$$x' \in \textstyle\bigcap_{1 \leq i \leq k} \{y \in \ell^\infty(\mathbb{N}) : |x_i^*(y - x')| < 1\} \subseteq U. \qquad (*)$$

We now apply Lemma 1.5.20 consecutively to the functionals $x_1^*, x_2^*, \ldots, x_k^*$ to arrive at an infinite subset L of $(\mathbb{N} \setminus M)$ such that $|x_i^*(z)| < 1$ whenever $1 \leq i \leq k$, $\|z\|_\infty \leq 2$ and $\text{supp}(z) \subseteq L$. Let $M' := (\mathbb{N} \setminus L)$. Then $M' \in P_\infty(\mathbb{N})$ and $M \subseteq M'$. We claim that $S(x', M') \subseteq U \cap S(x, M)$. Firstly, we will show that $S(x', M') \subseteq S(x, M)$. To this end, let $y \in S(x', M')$. Then $y_j = x'_j$ for all $j \in M'$. In particular, $y_j = x'_j$ for all $j \in M$, as $M \subseteq M'$. On the other hand, $x'_j = x_j$ for all $j \in M$, as $x' \in S(x, M)$. Therefore, $y_j = x'_j = x_j$ for all $j \in M$. Since $\|y\|_\infty \leq 1$, it follows that $y \in S(x, M)$. Thus, $S(x', M') \subseteq S(x, M)$. We now show that $S(x', M') \subseteq U$. Let $y \in S(x', M')$. Then $\|y\|_\infty \leq 1$ and $\text{supp}(y - x') \subseteq (\mathbb{N} \setminus M') = L$. Let $z := y - x'$. Then, $\|z\|_\infty \leq 2$ and $\text{supp}(z) \subseteq L$. Therefore, $|x_i^*(y - x')| = |x_i^*(z)| < 1$ for all $1 \leq i \leq k$. Hence, by the set-inclusion $(*)$, $y \in U$. □

Example 1.5.22 ([32, 37])**.** *$\beta(\mathbb{N}) \notin \mathcal{T}^*$.*

Proof. By Theorem 1.5.14 it is sufficient to show that the player Σ has a winning strategy in the $\mathcal{G}(\tau_p(\beta(\mathbb{N})), \|\cdot\|_\infty)$-game played on $(\beta(\mathbb{N}), \tau_p(\beta(\mathbb{N})), \|\cdot\|_\infty)$.

Let $\pi : \ell^\infty \to C(\beta(\mathbb{N}))$ be defined by, $\pi(x) := \tilde{x}$, where $\tilde{x}$ is the unique continuous extension of x, from $\mathbb{N}$ to $\beta(\mathbb{N})$. Note that π is linear and $\|\pi(x)\|_\infty = \|x\|_\infty$ for all $x \in \ell^\infty(\mathbb{N})$.

We now inductively define a winning strategy $t := (t_n : n \in \mathbb{N})$ for the player Σ. Let $x_\emptyset \in \ell^\infty(\mathbb{N})$ be defined by, $[x_\emptyset]_j = 1$ for all $j \in \mathbb{N}$ and let $M_\emptyset \in P_\infty(\mathbb{N})$ be defined by, $M_\emptyset := \varnothing$. Then let, $t_1(\emptyset) := \pi(S(x_\emptyset, M_\emptyset))$.

Step 1. Let (B_1) be a partial t-play of length 1. Then B_1 is a nonempty relatively $\tau_p(\beta(\mathbb{N}))$-open subset of

$$t_1(\emptyset) := \pi(S(x_\emptyset, M_\emptyset)).$$

Therefore, there exists a $\tau_p(\beta(\mathbb{N}))$-open, and hence weak-open, subset U of $C(\beta(\mathbb{N}))$ such that $B_1 = U \cap \pi(S(x_\emptyset, M_\emptyset))$. Then $\pi^{-1}(U)$ is a weak-open subset of $\ell^\infty(\mathbb{N})$ and $\pi^{-1}(U) \cap S(x_\emptyset, M_\emptyset) \neq \varnothing$. Hence, by Lemma 1.5.21 there exists $x_{(B_1)} \in \pi^{-1}(U) \cap S(x_\emptyset, M_\emptyset)$ and $M_{(B_1)} \in P_\infty(\mathbb{N})$ such that $M_\emptyset \subseteq M_{(B_1)}$ and

$$S(x_{(B_1)}, M_{(B_1)}) \subseteq \pi^{-1}(U) \cap S(x_\emptyset, M_\emptyset).$$

Let $t_2(B_1) := \pi(S(x_{(B_1)}, M_{(B_1)}))$.

Let $n \in \mathbb{N}$ and suppose that t_{j+1}, $x_{(B_1,\ldots,B_j)} \in \ell^\infty(\mathbb{N})$, and $M_{(B_1,\ldots,B_j)} \in P_\infty(\mathbb{N})$ have been defined for each partial t-play $(B_1, \ldots, B_j)$ of length j with $1 \leq j \leq n$ so that:

(i) $x_{(B_1,\ldots,B_j)} \in S(x_{(B_1,\ldots,B_{j-1})}, M_{(B_1,\ldots,B_{j-1})})$, here $(B_1, \ldots B_0)$ denotes $\emptyset$;

(ii) $M_{(B_1,\ldots,B_{j-1})} \subseteq M_{(B_1,\ldots,B_j)}$ [and so $S(x_{(B_1,\ldots,B_j)}, M_{(B_1,\ldots,B_j)}) \subseteq S(x_{(B_1,\ldots,B_{j-1})}, M_{(B_1,\ldots,B_{j-1})})$];

(iii) $t_{j+1}(B_1, \ldots, B_j) := \pi(S(x_{(B_1,\ldots,B_j)}, M_{(B_1,\ldots,B_j)}))$.

Step $n+1$. Let $(B_1, \ldots, B_{n+1})$ be a partial t-play of length $n+1$. Then B_{n+1} is a nonempty relatively $\tau_p(\beta(\mathbb{N}))$-open subset of

$$t_{n+1}(B_1, \ldots, B_n) := \pi(S(x_{(B_1,\ldots,B_n)}, M_{(B_1,\ldots,B_n)})).$$

Therefore, there exists a $\tau_p(\beta(\mathbb{N}))$-open, and hence weak-open, subset U of $C(\beta(\mathbb{N}))$ such that $B_{n+1} = U \cap \pi(S(x_{(B_1,\ldots,B_n)}, M_{(B_1,\ldots,B_n)}))$. Then $\pi^{-1}(U)$ is a weak-open subset of $\ell^\infty(\mathbb{N})$ and $\pi^{-1}(U) \cap S(x_{(B_1,\ldots,B_n)}, M_{(B_1,\ldots,B_n)}) \neq \varnothing$. Hence, by Lemma 1.5.21 there exists $x_{(B_1,\ldots,B_{n+1})} \in \pi^{-1}(U) \cap S(x_{(B_1,\ldots,B_n)}, M_{(B_1,\ldots,B_n)})$ and $M_{(B_1,\ldots,B_{n+1})} \in P_\infty(\mathbb{N})$ such that $M_{(B_1,\ldots,B_n)} \subseteq M_{(B_1,\ldots,B_{n+1})}$ and

$$S(x_{(B_1,\ldots,B_{n+1})}, M_{(B_1,\ldots,B_{n+1})}) \subseteq \pi^{-1}(U) \cap S(x_{(B_1,\ldots,B_n)}, M_{(B_1,\ldots,B_n)}).$$

Let $t_{(n+1)+1}(B_1, \ldots, B_{n+1}) := \pi(S(x_{(B_1,\ldots,B_{n+1})}, M_{(B_1,\ldots,B_{n+1})}))$. Then $t_{(n+1)+1}(B_1, \ldots, B_{n+1})$ is a nonempty subset of B_{n+1}. This completes the definition of $t := (t_n : n \in \mathbb{N})$.

It remains to show that t is a winning strategy for the player Σ. To this end, let $(B_n : n \in \mathbb{N})$ be an arbitrary t-play. Let $x : \mathbb{N} \to \mathbb{R}$ be defined by, $x_n := [x_{(B_1,\ldots,B_m)}]_n$ if $n \in \bigcup_{k\in\mathbb{N}} M_{(B_1,\ldots,B_k)}$ and m is the smallest natural number such that $n \in M_{(B_1,\ldots,B_m)}$. If $n \notin \bigcup_{k\in\mathbb{N}} M_{(B_1,\ldots,B_k)}$ then let $x_n := 0$. We claim that $x \in \bigcap_{n\in\mathbb{N}} S(x_{(B_1,\ldots,B_n)}, M_{(B_1,\ldots,B_n)})$. To substantiate this claim let us fix an $n \in \mathbb{N}$. We will show that $x \in S(x_{(B_1,\ldots,B_n)}, M_{(B_1,\ldots,B_n)})$. To this end, fix $j \in M_{(B_1,\ldots,B_n)}$. Let $m := \min\{k \in \mathbb{N} : j \in M_{(B_1,\ldots,B_k)}\}$. Then $1 \leq m \leq n$ and $j \in M_{(B_1,\ldots,B_m)}$. Therefore, by the definition of x, $x_j = [x_{(B_1,\ldots,B_m)}]_j$. On the other hand, since $m \leq n$

$$S(x_{(B_1,\ldots,B_n)}, M_{(B_1,\ldots,B_n)}) \subseteq S(x_{(B_1,\ldots,B_m)}, M_{(B_1,\ldots,B_m)}).$$

In particular, $x_{(B_1,\ldots,B_n)} \in S(x_{(B_1,\ldots,B_m)}, M_{(B_1,\ldots,B_m)})$ and so

$$x_j = [x_{(B_1,\ldots,B_m)}]_j = [x_{(B_1,\ldots,B_n)}]_j \quad \text{as } j \in M_{(B_1,\ldots,B_m)}.$$

Since $j \in M_{(B_1,\ldots,B_n)}$ was arbitrary, $x \in S(x_{(B_1,\ldots,B_n)}, M_{(B_1,\ldots,B_n)})$. Furthermore, since $n \in \mathbb{N}$ was arbitrary too, $x \in \bigcap_{n\in\mathbb{N}} S(x_{(B_1,\ldots,B_n)}, M_{(B_1,\ldots,B_n)})$. Therefore, $\pi(x) \in \bigcap_{n\in\mathbb{N}} B_n$ and so $\bigcap_{n\in\mathbb{N}} B_n \neq \varnothing$. [Note: one could also, perhaps more easily argue, that each set $S(x_{(B_1,\ldots,B_n)}, M_{(B_1,\ldots,B_n)})$ is weak* compact and then deduce from the finite intersection property that $\bigcap_{n\in\mathbb{N}} S(x_{(B_1,\ldots,B_n)}, M_{(B_1,\ldots,B_n)}) \neq \varnothing$.] Finally, let us recall that

$$\|\cdot\|_\infty - \text{diam}[\pi(S(x_{(B_1,\ldots,B_n)}, M_{(B_1,\ldots,B_n)}))] = \|\cdot\|_\infty - \text{diam}[S(x_{(B_1,\ldots,B_n)}, M_{(B_1,\ldots,B_n)})] = 2$$

for all $n \in \mathbb{N}$. Thus, $\|\cdot\|_\infty - \text{diam}[B_n] = 2$ for all $n \in \mathbb{N}$; which implies that Σ wins the play $(B_n : n \in \mathbb{N})$. □

1.6 Characterisation of co-Namioka spaces

In this section, we will follow the model of the last section, to obtain a characterisation for the class of co-Namioka spaces.

Exercise 1.6.1. *Let (X, τ) and (Y, τ') be topological spaces. Show that if $f : X \to Y$ is a quasicontinuous surjection and D is a dense subset of (X, τ), then $f(D)$ is a dense subset of (Y, τ).*

Proposition 1.6.2. *Let $f : X \to Y$ be a quasicontinuous, open mapping from a topological space (X, τ) onto a topological space (Y, τ'). If (X, τ) is a Baire space, then so is (Y, τ')*

Proof. Let $(O_n : n \in \mathbb{N})$ be a countable family of dense open subsets of (Y, τ'). We will show that $\bigcap_{n\in\mathbb{N}} O_n$ is dense in (Y, τ). For each $n \in \mathbb{N}$, let $U := \text{int}(f^{-1}(O_n))$. Then, clearly, each set U_n is open in (X, τ). We will now show that each set U_n is also dense in (X, τ). To this end, let $n \in \mathbb{N}$ and let W be a nonempty open subset of (X, τ). Since f is an open mapping, $f(W)$ is a nonempty open subset of (Y, τ'). Furthermore, since O_n is dense in (Y, τ'), $f(W) \cap O_n \neq \varnothing$. Choose $x_0 \in W$ such that $f(x_0) \in O_n$. It now follows from the fact that f is quasicontinuous at x_0 that there exists a nonempty open subset V of W such that $f(V) \subseteq O_n$. Therefore, $V \subseteq \text{int}(f^{-1}(O_n))$. Hence, $\varnothing \neq V \subseteq W \cap \text{int}(f^{-1}(O_n)) = W \cap U_n$. Since (X, τ) is a Baire space $\bigcap_{n\in\mathbb{N}} U_n$ is dense in (X, τ). Therefore, by Exercise 1.6.1, $f(\bigcap_{n\in\mathbb{N}} U_n)$ is dense in (Y, τ'). It now only remains to observe that

$$f(\textstyle\bigcap_{n\in\mathbb{N}} U_n) \subseteq \bigcap_{n\in\mathbb{N}} f(U_n) \subseteq \bigcap_{n\in\mathbb{N}} O_n.$$

This completes the proof. □

Proposition 1.6.3. *Let $f : X \to Y$ be a quasicontinuous mapping from a topological space (X, τ) onto a topological space (Y, τ'). If (X, τ) is a Baire space, then so is the graph of f, $\text{Gr}(f)$, endowed with the relative product topology of (X, τ) and (Y, τ').*

Proof. The proof of this result is identical to the proof of Proposition 1.5.2. Hence we leave it as an exercise for the reader. □

Proposition 1.6.4. *Let (Y, τ') be a topological space and let d be some metric defined on it. Then the following conditions are equivalent:*

(i) *every τ'-continuous mapping $f : X \to Y$ from a Baire space (X, τ) into (Y, τ') is d-continuous at the points of a dense subset of (X, τ);*

(ii) *every τ'-quasicontinuous mapping $f : X \to Y$ from a Baire space (X, τ) into (Y, τ'), is d-continuous at the points of a dense subset of (X, τ).*

Proof. The proof of this result is identical to the proof of Proposition 1.5.3. Hence we leave it as an exercise for the reader. □

Corollary 1.6.5. *Let (Y, τ') be a nonempty compact Hausdorff topological space and let X be a nonempty subset of $C(Y)$. Then the following conditions are equivalent:*

(i) *every $\tau_p(Y)$-continuous mapping $f : Z \to X$ from a Baire space (Z, τ) into $C(Y)$ is $\|\cdot\|_\infty$-continuous at the points of a dense subset of (Z, τ);*

(ii) *every $\tau_p(Y)$-quasicontinuous mapping $f : Z \to X$ from a Baire space (Z, τ) into X, is $\|\cdot\|_\infty$-continuous at the points of a dense subset of (Z, τ).*

Theorem 1.6.6. *Let (X, τ) be a topological space, let*

$$P := \{(A_n : n \in \mathbb{N}) \in \tau^{\mathbb{N}} : \varnothing \neq A_{n+1} \subseteq A_n \text{ for all } n \in \mathbb{N}\}$$

and let $d : P \times P \to [0, \infty)$ be the Baire metric on P. Then (P, d) is a nonempty complete metric space. If (X, τ) is a Baire space, then $S := \{(A_n : n \in \mathbb{N}) \in P : \bigcap_{n\in\mathbb{N}} A_n \neq \varnothing\}$ is everywhere second category in (P, d).

Proof. Firstly, we will show that (P, d) is a nonempty complete metric space. The fact that P is nonempty is obvious and the proof that the Baire metric d is indeed a metric follows from Lemma 1.5.5. So we need to show that (P, d) is complete. To do this end, let $(p^n : n \in \mathbb{N})$ be a Cauchy sequence in (P, d). For each $n \in \mathbb{N}$, let $p^n := (A^n_k : k \in \mathbb{N})$. Since $(p^n : n \in \mathbb{N})$ be a Cauchy sequence there exists a strictly increasing sequence $(n_k : k \in \mathbb{N})$ of natural numbers such that $d - \text{diam}[\{p^i : n_k \leq i\}] < 1/k$ for all $k \in \mathbb{N}$. In particular,

$$d(p^{n_k}, p^i) < 1/k \quad \text{for all } k \in \mathbb{N} \text{ and all } n_k \leq i.$$

Therefore,

$$A^{n_k}_j = A^i_j \quad \text{for all } k \in \mathbb{N}, \text{ all } 1 \leq j \leq k \text{ and all } n_k \leq i.$$

By setting $j := k$ we get that

$$A^{n_k}_k = A^i_k \quad \text{for all } k \in \mathbb{N} \text{ and all } n_k \leq i. \qquad (*)$$

We now define a new sequence $p := (A_k : k \in \mathbb{N})$ by, $A_k := A^{n_k}_k$ for all $k \in \mathbb{N}$. Notice that for each fixed $k \in \mathbb{N}$, it follows from the definition of p and Equation $(*)$ (with k replaced by j and i replaced by n_k) that

$$A_j = A^{n_j}_j = A^{n_k}_j \quad \text{for all } 1 \leq j \leq k \qquad (**)$$

since $n_j \leq n_k$. Now, for each $2 \leq k \in \mathbb{N}$,

$$A_k = A_k^{n_k} \subseteq A_{k-1}^{n_k} = A_{k-1}, \quad \text{by substituting } j = k-1 \text{ into Equation } (**)$$

Therefore, $p \in P$. Finally, note that $d(p^i, p) < 1/k$ for all $n_k \leq i$. To see this, fix $k \in \mathbb{N}$ and let $1 \leq j \leq k$. Then, by the definition of p and Equation $(*)$, (with k replaced by j) we have that

$$A_j = A_j^{n_j} = A_j^i \quad \text{for all } n_j \leq i.$$

Since $n_j \leq n_k$ it follows that $A_j = A_j^i$ for all $n_k \leq i$. This show that $(p^n : n \in \mathbb{N})$ converges to $p \in P$.

We now show that S is everywhere second category in (P, d).

Suppose, for the purpose of obtaining a contradiction, that S is not everywhere second category in (P, d). Then there exists a nonempty open subset A of (P, d) such that $S \cap A$ is a first category subset of (P, d). Therefore, there exists a sequence $(F_n : n \in \mathbb{N})$ of closed, nowhere dense subsets of (P, d), such that $S \cap A \subseteq \bigcup_{n \in \mathbb{N}} F_n$.

For each $n \in \mathbb{N}$, let $O_n := A \setminus F_n$. Then each set O_n is open and dense in A. Furthermore, $(\bigcap_{n \in \mathbb{N}} O_n) \cap S = \varnothing$. We now inductively define a (necessarily non-winning) strategy $t := (t_n : n \in \mathbb{N})$ for the player β in the Choquet-game played on (X, τ).

Base Step. Choose $p_\emptyset \in P$ and $m_\emptyset \in \mathbb{N}$ so that $B_d[p_\emptyset, 1/m_\emptyset] \subseteq O_1 \cap A$. Note that this is possible since O_1 is dense and open in A. Then define, $t_1(\emptyset) := I(p_\emptyset, m_\emptyset)$, where $I : P \times \mathbb{N} \to \tau$ is defined by, $I((A_k : k \in \mathbb{N}), n) := A_n$, for all $((A_k : k \in \mathbb{N}), n) \in P \times \mathbb{N}$.

Step 1. Suppose that (A_1) is a partial t-play of length 1. Then $A_1 \subseteq t_1(\emptyset) = I(p_\emptyset, m_\emptyset)$. Let $p^* := (A_n^* : n \in \mathbb{N})$ be defined by, $A_j^* := I(p_\emptyset, j)$ for all $1 \leq j \leq m_\emptyset$ and $A_j^* := A_1$ for all $m_\emptyset < j$. Then $p^* \in P$ and moreover, $B_d[p^*, 1/(m_\emptyset + 1)] \subseteq B_d(p_\emptyset, 1/m_\emptyset)$. Now, $B_d(p^*, 1/(m_\emptyset + 1)) \cap O_2 \neq \varnothing$. Therefore, there exists a $p_{(A_1)} \in P$ and an $m_{(A_1)} \in \mathbb{N}$ such that $m_\emptyset < m_{(A_1)}$ and

$$B_d[p_{(A_1)}, 1/m_{(A_1)}] \subseteq B_d(p^*, 1/(m_\emptyset + 1)) \cap O_2 \subseteq B_d(p_\emptyset, 1/m_\emptyset) \cap O_2.$$

We then define, $t_2(A_1) := I(p_{(A_1)}, m_{(A_1)}) \subseteq A_1$.

Now, let $n \in \mathbb{N}$ and suppose that $p_{(A_1,\ldots,A_j)} \in P$, $m_{(A_1,\ldots,A_j)} \in \mathbb{N}$ and t_j have been defined for every $1 \leq j \leq n$ so that:

(i) $m_{(A_1,\ldots,A_{j-1})} < m_{(A_1,\ldots,A_j)}$, where $(A_1, \ldots A_0) := \emptyset$;

(ii) $B_d[p_{(A_1,\ldots,A_j)}, 1/m_{(A_1,\ldots,A_j)}] \subseteq O_{j+1} \cap B_d(p_{(A_1,\ldots,A_{j-1})}, 1/m_{(A_1,\ldots,A_{j-1})}) \subseteq O_1$;

(iii) $t_{j+1}(A_1, \ldots, A_j) := I(p_{(A_1,\ldots,A_j)}, m_{(A_1,\ldots,A_j)}) \subseteq A_j$.

Step $n+1$. Suppose that $(A_1, \ldots, A_{n+1})$ is a partial t-play of length $n+1$ then

$$A_{n+1} \subseteq I(p_{(A_1,\ldots,A_n)}, m_{(A_1,\ldots,A_n)}) \subseteq A_n.$$

Let $p^* := (A_n^* : n \in \mathbb{N})$ be defined by, $A_j^* := I(p_{(A_1,\ldots,A_n)}, j)$ for all $1 \leq j \leq m_{(A_1,\ldots,A_n)}$ and $A_j^* := A_{n+1}$ for all $m_{(A_1,\ldots,A_n)} < j$. Then $p^* \in P$ and moreover,

$$B_d[p^*, 1/(m_{(A_1,\ldots,A_n)} + 1)] \subseteq B_d(p_{(A_1,\ldots,A_n)}, 1/m_{(A_1,\ldots,A_n)}).$$

Now, $B_d(p^*, 1/(m_{(A_1,\ldots,A_n)}+1)) \cap O_{n+2} \neq \varnothing$. Therefore, there exists a $p_{(A,\ldots,A_{n+1})} \in P$ and an $m_{(A_1,\ldots,A_{n+1})} \in \mathbb{N}$ such that $m_{(A_1,\ldots,A_n)} < m_{(A_1,\ldots,A_{n+1})}$ and

$$\begin{aligned} B_d[(p_{(A_1,\ldots,A_{n+1})}, 1/m_{(A_1,\ldots,A_{n+1})}] &\subseteq B_d(p^*, 1/(m_{(A_1,\ldots,A_n)}+1)) \cap O_{n+2} \\ &\subseteq B_d(p_{(A_1,\ldots,A_n)}, 1/m_{(A_1,\ldots,A_n)}) \cap O_{n+2}. \end{aligned}$$

We then define, $t_{(n+1)+1}(A_1, \ldots, A_{n+1}) := I(p_{(A_1,\ldots,A_{n+1})}, m_{(A_1,\ldots,A_{n+1})}) \subseteq A_{(n+1)+1}$.

This completes the definition of $t := (t_n : n \in \mathbb{N})$. Since (X, τ) is a Baire space, we have, by Theorem 1.1.5, that t is not a winning strategy for the player β. Therefore, there exists a t-play $(A_n : n \in \mathbb{N})$ where α wins, i.e., where $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$.

Now, since $m_{(A_1,\ldots,A_n)} < m_{(A_1,\ldots,A_{n+1})}$ for all $n \in \mathbb{N}$, it follows that $n \leq m_{(A_1,\ldots,A_n)}$ for all $n \in \mathbb{N}$. Let $\{p^*\} := \bigcap_{n\in\mathbb{N}} B_d[p_{(A_1,\ldots,A_n)}, 1/m_{(A_1,\ldots,A_n)}]$. This is well-defined, by the Cantor intersection property, since (P, d) is a complete metric space. Moreover, it follows from Property (ii) that $p^* \in \bigcap_{n\in\mathbb{N}} O_n$.

So, to obtain the desired contradiction, we nee only show that $p^* \in S$. To this end, consider the following.

$$\begin{aligned} \varnothing \neq \textstyle\bigcap_{n\in\mathbb{N}} A_n &= \textstyle\bigcap_{n\in\mathbb{N}} A_{n+1} \\ &\subseteq \textstyle\bigcap_{n\in\mathbb{N}} t_{n+1}(A_1, \ldots, A_n) \\ &= \textstyle\bigcap_{n\in\mathbb{N}} I(p_{(A_1,\ldots,A_n)}, m_{(A_1,\ldots,A_n)}) \\ &= \textstyle\bigcap_{n\in\mathbb{N}} I(p^*, m_{(A_1,\ldots,A_n)}) = \bigcap_{n\in\mathbb{N}} I(p^*, n). \end{aligned}$$

This completes the proof. □

Lemma 1.6.7. *Let (X, τ) be a Baire space, let*

$$P := \{(A_k : k \in \mathbb{N}) \in \tau^{\mathbb{N}} : \varnothing \neq A_{k+1} \subseteq A_k \text{ for all } k \in \mathbb{N}\}$$

and let $d : P \times P \to [0, \infty)$ be the Baire metric on P. If $f : X \to Y$ is a τ'-quasicontinuous function into a topological space (Y, τ'), then $F : P \to 2^Y$ defined by, $F(p) := \bigcap_{k\in\mathbb{N}} f(A_k)$, where $p := (A_k : k \in \mathbb{N})$, for all $p \in P$, is a τ'-minimal mapping. Moreover, F has nonempty images on the set $S := \{(A_k : k \in \mathbb{N}) \in P : \bigcap_{k\in\mathbb{N}} A_k \neq \varnothing\}$ and for each $p \in P$, $n \in \mathbb{N}$ and $A \subseteq Y$ with $F(B_d(p, 1/n)) \subseteq A$, $f(A_{n+1}) \subseteq \overline{A}^{\tau'}$, where $p := (A_k : k \in \mathbb{N})$.

Proof. It is clear that F has nonempty images on the set S. Next we show that F is τ'-minimal. To accomplish this, let us consider an open subset U of (P, d) and a τ'-open subset W of Y such that $F(U) \cap W \neq \varnothing$. Choose $p := (A_k : k \in \mathbb{N}) \in U$ such that $F(p) \cap W \neq \varnothing$. Then $f(A_k) \cap W \neq \varnothing$ for all $k \in \mathbb{N}$. Furthermore, as U is open in (P, d), there exists an $n \in \mathbb{N}$ such that $B_d(p, 1/n) \subseteq U$. Since f is quasicontinuous and $f(A_n) \cap W \neq \varnothing$ there exists a nonempty open subset V' of A_n such that $f(V') \subseteq W$.

Let $p^* := (A_k^* : k \in \mathbb{N})$ be defined by, $A_k^* := A_k$ for all $1 \leq k \leq n$ and $A_k^* := V'$ for all $n < k \in \mathbb{N}$. Then $p^* \in P$. In fact, $p^* \in B_d(p, 1/n)$. Furthermore, if we set $V := B_d(p^*, 1/(n+1))$, then V is a nonempty open subset of $B_d(p, 1/n) \subseteq U$ and $F(V) \subseteq W$. To see this last set-inclusion, consider $p' := (A'_k : k \in \mathbb{N}) \in V$. Then $A'_{n+1} = A^*_{n+1} = V'$ and so

$$F(p') = \textstyle\bigcap_{k\in\mathbb{N}} f(A'_k) \subseteq f(A'_{n+1}) = f(A^*_{n+1}) = f(V') \subseteq W.$$

This shows that F is a τ'-minimal mapping.

To verify the last assertion of the lemma, we consider $p := (A_k : k \in \mathbb{N}) \in P$, $n \in \mathbb{N}$ and $A \subseteq Y$ such that $F(B_d(p, 1/n)) \subseteq A$ and suppose, in order to obtain a contradiction, that $f(A_{n+1}) \not\subseteq \overline{A}^{\tau'}$.

Since f is τ-quasicontinuous there exists a nonempty open subset V of A_{n+1} such that $f(V) \cap \overline{A}^{\tau'} = \varnothing$. Let $p' := (A'_k : k \in \mathbb{N})$ be defined by, $A'_k := A_k$ of $1 \leq k \leq n$ and $A'_k := V$ if $n < k \in \mathbb{N}$. Then $p' \in B_d(p, 1/n)$, but

$$\varnothing \neq f(V) = \bigcap_{k \in \mathbb{N}} f(A_k) = F(p') \subseteq Y \setminus \overline{A}^{\tau'} \subseteq Y \setminus A;$$

which contradicts the assumption that $\varnothing \neq F(p') \subseteq F(B_d(p, 1/n)) \subseteq A$. □

Theorem 1.6.8. *Let (Y, τ, τ') be a bitopological space with the property that:*

(a) *(Y, τ') is T_1 and*

(b) *for every $U \in \tau'$ and every $x \in U$ there exists a $V \in \tau'$ such that $x \in V \subseteq \overline{V}^{\tau} \subseteq U$.*

Then the following are equivalent:

(i) *every τ-quasicontinuous mapping $f : X \to Y$ from a Baire metric space (X, ρ) into Y has at least one point of τ'-continuity;*

(ii) *every τ-quasicontinuous mapping $f : X \to Y$ from a Baire space (X, τ'') into Y has at least one point of τ'-continuity;*

(iii) *every τ-quasicontinuous mapping $f : X \to Y$ from a Baire space (X, τ'') into Y is τ'-continuous at each point of a residual subset of (X, τ'')*

(iv) *every τ-quasicontinuous mapping $f : X \to Y$ from a Baire metric space (X, ρ) into Y is τ'-continuous at each point of a residual subset of (X, τ'').*

Proof. $(i) \Rightarrow (ii)$. Suppose that (i) holds. Let $f : X \to Y$ be a τ-quasicontinuous mapping from a Baire space (X, τ'') into Y. Let

$$P := \{(A_k : k \in \mathbb{N}) \in \tau^{\mathbb{N}} : \varnothing \neq A_{k+1} \subseteq A_k \text{ for all } k \in \mathbb{N}\}$$

and let $F : P \to 2^Y$ be defined by, $F(p) := \bigcap_{k \in \mathbb{N}} f(A_k)$, where $p := (A_k : k \in \mathbb{N})$, for all $p \in P$. By Theorem 1.6.6 and Lemma 1.6.7 we see that F is a τ-minimal mapping that has nonempty images on the set $S := \{(A_k : k \in \mathbb{N}) \in P : \bigcap_{k \in \mathbb{N}} A_k \neq \varnothing\}$. Furthermore, by Theorem 1.6.6, S is an everywhere second category subset of (P, d). By Exercise 1.5.6, $F|_S : S \to 2^Y$, defined by, $F|_S(p) := F(p)$ for all $p \in S$, is also a τ-minimal mapping. Let $d_S : S \times S \to [0, \infty)$ denote the restriction of d to S. Then (S, d_S) is a Baire metric space.

Let $s : S \to Y$ be any selection of $F|_S$. Then, by Exercises 1.5.6 and 1.5.6, it follows that s is τ-quasicontinuous on S. Therefore, by assumption, there exists a $p := (A_k : k \in \mathbb{N}) \in S$ such that s is τ'-continuous at p. Now, by Proposition 1.5.11, $F(p) = \{s(p)\}$ and F is τ'-upper semicontinuous at p.

We now claim that f is τ'-continuous at each point of $\bigcap_{k\in\mathbb{N}} A_k$; which is nonempty as $p \in S$. To accomplish this, we consider an arbitrary element $x \in \bigcap_{k\in\mathbb{N}} A_k$. Let U be an arbitrary τ'-open neighbourhood of $f(x)$. Then, by assumption, there exists a τ'-open neighbourhood V of $f(x)$ such that

$$s(p) = \{f(x)\} \subseteq V \subseteq \overline{V}^{\tau} \subseteq U.$$

Since F is τ'-upper semicontinuous at p there exists an $n \in \mathbb{N}$ such that $F(B_d(p, 1/n)) \subseteq V$. Then, by Lemma 1.6.7, $f(A_{n+1}) \subseteq \overline{V}^{\tau} \subseteq U$. Since $x \in A_{n+1}$, this shows that f is τ'-continuous at x.

$(ii) \Rightarrow (iii)$. Assume that (ii) holds. Let $f : X \to Y$ be a τ-quasicontinuous mapping from a Baire space (X, τ'') and suppose, for the purpose of obtaining a contradiction that $\{x \in X : f \text{ is } \tau'\text{-continuous at } x\}$ is not residual in (X, τ''). Then

$$A := \{x \in X : f \text{ is not } \tau'\text{-continuous at } x\}$$

is a second category subset of (X, τ''). Therefore, by Proposition 1.6.14 there exists a nonempty open subset U of (X, τ'') such that A is everywhere second category in U. In particular, see page 67, $S := A \cap U$, with the relative topology, is a Baire space. Now, by Proposition 1.6.16 part (i), $f|_U : U \to Y$ is a τ-quasicontinuous mapping, and further, by Proposition 1.6.16 part (ii), $(f|_U)|_S : S \to Y$ is also τ-quasicontinuous on S. Note of course, that $(f|_U)|_S = f|_S$. Therefore, it follows that $f|_S : S \to Y$, has a point $x \in S$, where $f|_S$ is τ'-continuous. We may now apply Proposition 1.5.11 part (ii), in conjunction with Exercise 1.5.8, to deduce that $f|_U$ is τ'-continuous at x. Finally, since U is an open set, it follows that f is τ'-continuous at $x \in S \subseteq A$; which contradicts the definition of the set A. Thus, we have obtained our desired contradiction, which in turn means that Property (iii) holds.

$(iii) \Rightarrow (iv)$ and $(iv) \Rightarrow (i)$ are obvious. □

Proposition 1.6.9. *Let (X, d) be a Baire metric space and let (Y, τ') be a nonempty compact Hausdorff topological space. If $f : X \to C(Y)$ is a $\tau_p(Y)$-quasicontinuous mapping and $\{x \in X : f \text{ is } \tau_p(Y)\text{-continuous at } x\}$ is residual in (X, d), then the set $\{x \in X : f \text{ is } \|\cdot\|_\infty\text{-continuous at } x\}$ is dense in (X, d).*

Proof. Suppose that $f : X \to C(Y)$ is a $\tau_p(Y)$-quasicontinuous mapping from a Baire metric space (X, d) into $C(Y)$, for some nonempty compact Hausdorff topological space (Y, τ'). Suppose further, that $R := \{x \in X : f \text{ is } \tau_p(Y)\text{-continuous at } x\}$ is residual in (X, d). Then (R, d_R) is also a Baire metric space, where $d_R : R \times R \to [0, \infty)$ denotes the restriction of d to the set R.

Now, $f|_R : R \to C(Y)$, defined by, $f|_R(x) := f(x)$ for all $x \in R$, is a $\tau_p(Y)$-continuous mapping. Therefore, by Exercise 1.2.13 part (i) and Theorem 1.2.14, there exists a dense subset D of R such that $f|_R$ is continuous, with respect to the supremum norm topology on $C(Y)$, at each point of D.

Now, D is dense in (X, d) and f is continuous, with respect to the supremum norm topology on $C(Y)$, at each point of D. To justify this latter fact, let us consider an arbitrary point

$x_0 \in D$ and any positive real number ε. Since $f|_R$ is continuous, with respect to the supremum norm topology on $C(Y)$, at x_0, there exists an open subset U of (X, d) such that

$$f|_R(U \cap R) \subseteq B(f|_R(x_0), \varepsilon) = B(f(x_0), \varepsilon) \subseteq B[f(x_0), \varepsilon].$$

Now, since $B_{C(Y)}$ is $\tau_p(Y)$-closed, $B[f(x_0), \varepsilon] = f(x_0) + \varepsilon B_{C(Y)}$ is $\tau_p(Y)$-closed. Therefore, from the $\tau_p(Y)$-quasicontinuity of f, it follows that $f(U) \subseteq B[f(x_0), \varepsilon]$. Since $x_0 \in D$ and $0 < \varepsilon$ were both arbitrary, it follows that f is continuous, with respect to the supremum norm topology on $C(Y)$, at each point of D. □

We now present a characterisation of co-Namioka spaces.

Theorem 1.6.10. *Let (Y, τ') be a nonempty compact Hausdorff topological space. Then the following conditions are equivalent:*

(i) *every $\tau_p(Y)$-quasicontinuous mapping $f : Z \to C(Y)$ from a Baire metric space (Z, ρ), has at least one point of $\tau_p(Y)$-continuity;*

(ii) *every $\tau_p(Y)$-quasicontinuous mapping $f : Z \to C(Y)$ from a Baire metric space (Z, ρ), is $\tau_p(Y)$-continuous at each point of a residual subset of (Z, ρ);*

(iii) *every $\tau_p(Y)$-quasicontinuous mapping $f : Z \to C(Y)$ from a Baire metric space (Z, ρ), has at least one point of $\|\cdot\|_\infty$-continuity;*

(iv) *every $\tau_p(Y)$-quasicontinuous mapping $f : Z \to C(Y)$ from a Baire space (Z, τ), is $\|\cdot\|_\infty$-continuous at each point of a dense subset of (Z, τ);*

(v) *every $\tau_p(Y)$-continuous mapping $f : Z \to C(Y)$ from a Baire space (Z, τ), is $\|\cdot\|_\infty$-continuous at each point of a dense and G_δ subset of (Z, τ).*

Proof. *(i)* ⇒ *(ii)*. This follows directly from Theorem 1.6.8.

(ii) ⇒ *(iii)*. This follows directly from Proposition 1.6.9.

(iii) ⇒ *(iv)* follows directly from Theorem 1.6.8.

Now, *(iv)* ⇔ *(v)* follows from Corollary 1.6.5 and the fact that the set of points of norm continuity always form a G_δ set.

(iv) ⇒ *(i)* is obvious, after one recalls that all complete metric spaces are weakly α-favourable. □

We note that condition *(v)* in Theorem 1.6.10 is the definition of a co-Namioka space. Hence each one of the conditions in Theorem 1.6.10 provides an alternative description of the class of co-Namioka spaces.

We will now present some applications of Theorem 1.6.10.

Corollary 1.6.11. *Suppose that (Y_1, τ_1') and (Y_2, τ_2') are nonempty compact Hausdorff topological spaces. If $C_p(Y_1)$ is homeomorphic to $C_p(Y_2)$, then $(Y_1, \tau_1') \in \mathcal{N}^*$ if, and only if, $(Y_2, \tau_2') \in \mathcal{N}^*$.*

Proof. This holds because either conditions (i) or (ii) of Theorem 1.6.10 characterise the class $\mathcal{N}^*$ soley in terms of the $\tau_p(Y)$-topology. $\square$

Corollary 1.6.12 ([19]). *Let (Y, τ') be a nonempty compact Hausdorff topological space. If $\{Y_n : n \in \mathbb{N}\}$ is a cover of Y consisting of closed subsets and $Y_n \in \mathcal{N}^*$ for each $n \in \mathbb{N}$, then $(Y, \tau') \in \mathcal{N}^*$.*

Proof. Let $f : X \to C(Y)$ be a $\tau_p(Y)$-quasicontinuous function from a Baire metric space (X, d). For each $n \in \mathbb{N}$, let $f_n : X \to C(Y_n)$ be defined by, $f_n(x) := f(x)|_{Y_n}$ for each $x \in X$. Then each f_n is $\tau_p(Y_n)$-quasicontinuous on (X, d). For each $n \in \mathbb{N}$, let R_n denote the dense and G_δ subset of (X, d) where f_n is continuous with respect to the supremum norm topology on $C(Y_n)$. Let $R := \bigcap_{n\in\mathbb{N}} R_n$. Then R is a dense and G_δ subset of (X, d) and f is continuous with respect to the $\tau_p(Y)$-topology, at each point of R. The result now follows from Theorem 1.6.10. $\square$

The following theorem extends [38, A.2 Theorem].

Theorem 1.6.13. *Let (Y, τ') be a nonempty compact Hausdorff space. Then $(Y, \tau') \in \mathcal{N}^*$ if, for each completely regular Baire topological space (X, τ), with countable tightness, $N(X, Y)$ holds.*

Proof. We will appeal to Theorem 1.6.10. Let $f : X \to C(Y)$ be a $\tau_p(Y)$-quasicontinuous mapping from a Baire metric space (X, d). Then, the graph of f, $\mathrm{Gr}(f)$, endowed with the relative product topology of (X, d) and $(C(Y), \tau_p(Y))$ is a completely regular Baire space, see Proposition 1.6.3. Furthermore, since $(C(Y), \tau_p(Y))$ has countable tightness, see [3, Theorem II.1.1] and (X, d) is first countable, it follows from [17, Theorem 4.2] that $(X, d) \times (C(Y), \tau_p(Y))$ has countable tightness. Therefore, $\mathrm{Gr}(f)$, endowed with the relative product topology of (X, d) and $(C(Y), \tau_p(Y))$, has countable tightness.

Next, as in Proposition 1.5.3, it follows that f is continuous with respect to the supremum norm topology on $C(Y)$, at each point of a dense and G_δ subset of (X, d). It then follows from Theorem 1.6.10, that $(Y, \tau') \in \mathcal{N}^*$. $\square$

Preliminary Results

Proposition 1.6.14. *Let A be a subset of a topological space (X,τ), then $A \setminus U(A)$ is a first category set. In particular, if A is a second category set in (X,τ), then $U(A)$ is nonempty.*

Proof. Let $\mathcal{W} := \{W_\gamma : \gamma \in \Gamma\}$ be a maximal, with respect to set inclusion, family of pairwise disjoint nonempty open subsets of $X \setminus \overline{U(A)}^\tau$ with the property that $W_\gamma \cap A$ is a first category subset of (X,τ) for each $\gamma \in \Gamma$. Note that by Zorn's Lemma such a maximal family exists. We claim that $W := \bigcup_{\gamma\in\Gamma} W_\gamma$ is dense in $X \setminus \overline{U(A)}^\tau$. Indeed, if this is not the case then there exists a nonempty open subset V of $X \setminus \overline{U(A)}^\tau$ such that $V \cap W = \varnothing$. Since $V \not\subseteq U(A)$, A is not everywhere second category in V, that is, there exists a nonempty open subset $V' \subseteq V$ such that $V' \cap A$ is first category in (X,τ). Let $\Gamma' := \Gamma \cup \{\Gamma\}$, $W_\Gamma := V'$ and let $\mathcal{W}' := \{W_\gamma : \gamma \in \Gamma'\}$. Then $\mathcal{W}'$ strictly contains $\mathcal{W}$ and consists of pairwise disjoint nonempty open subsets of $X \setminus \overline{U(A)}^\tau$ with the property that $W_\gamma \cap A$ is a first category subset of (X,τ) for each $\gamma \in \Gamma'$. This contradicts the maximality of $\mathcal{W}$. Therefore, we may conclude that W is indeed dense in $X \setminus \overline{U(A)}^\tau$. Next we observe that

$$\begin{aligned} A \setminus U(A) &= \Big([A \setminus U(A)] \cap W\Big) \cup \Big([A \setminus U(A)] \setminus W\Big) \\ &= \Big([A \setminus U(A)] \cap W\Big) \cup \Big(A \setminus (U(A) \cup W)\Big) \quad \text{by De Morgan's laws} \\ &\subseteq \Big(A \cap W\Big) \cup \Big(X \setminus (U(A) \cup W)\Big). \end{aligned}$$

Now, it is not hard to verify that $U(A) \cup W$ is a dense open subset of X. Therefore, $X \setminus (U(A) \cup W)$ is a closed nowhere dense subset of X. Thus to show that $A \setminus U(A)$ is of the first Baire category it is sufficient to show that $A \cap W$ is of the first Baire category. For each $\gamma \in \Gamma$ there exists a countably family of closed subsets $\{F_\gamma^n : n \in \mathbb{N}\}$ of $\overline{W_\gamma}^\tau$ such that $A \cap W_\gamma \subseteq \bigcup_{n\in\mathbb{N}} F_\gamma^n$ and $\text{int}(F_\gamma^n) = \varnothing$. We define, for each $n \in \mathbb{N}$, $F^n := \bigcup\{F_\gamma^n : \gamma \in \Gamma\}$. It is readily checked that each F^n is nowhere dense in X. However, $A \cap W \subseteq \bigcup_{n\in\mathbb{N}} F^n$ and so we may conclude that $A \cap W$ is first category on (X,τ). □

Exercise 1.6.15. *Suppose that (X,τ) and (Y,τ') are topological spaces and $f : X \to Y$. Show that f is quasicontinuous on (X,τ) if, and only if, for each pair of open subsets U of X and W of Y such that $f(U) \cap W \neq \varnothing$, there exists a nonempty open subset V of U such that $f(V) \subseteq W$.*

Proposition 1.6.16. *Let $f : X \to Y$ be a quasicontinuous function acting from a topological space (X,τ) into a topological space (Y,τ'). If $\varnothing \neq A \subseteq X$ is: (i) open, or (ii) dense* *in (X,τ), then the restriction of f to A, denoted $f|_A : A \to Y$, is also a quasicontinuous mapping, (with of course, A endowed with the relative τ-topology).*

Proof. (i) Let A be a nonempty open subset of (X,τ). To show that $f|_A$ is quasicontinuous we shall appeal to Exercise 1.6.15. To this end, let U be an open subset of A and let W be an open subset of Y such that $f|_A(U) \cap W \neq \varnothing$. Since A is open in (X,τ) so is U. Moreover, since $f|_A(U) \cap W \neq \varnothing$, $f(U) \cap W \neq \varnothing$. Therefore, by Exercise 1.6.15, there exists a nonempty open subset V of U such that $f|_A(V) = f(V) \subseteq W$.

(ii) Suppose that A is a dense subset of (X,τ). To show that $f|_A$ is quasicontinuous we shall again appeal to Exercise 1.6.15. To this end, let U be an open subset of A and let W be an open subset of Y such that $f|_A(U) \cap W \neq \varnothing$. Then, let U' be an open subset of (X,τ) such that $U = U' \cap A$. Since

$$\varnothing \neq f|_A(U) \cap W \subseteq f(U') \cap W$$

it follows from Exercise 1.6.15, that there exists a nonempty open subset V' of U' such that $f(V') \subseteq W$. Let $V := V' \cap A$. Then V is a nonempty (since A is dense), open (in the relative τ-topology on A) subset of U and $f|_A(V) \subseteq f(V') \subseteq W$. □

We will call a topological space (X,τ) *pointwise countably complete*, if it is regular and there exists a sequence $(\mathscr{A}_n : n \in \mathbb{N})$ of open covers of X with the property that every monotonically decreasing sequence $(F_n : n \in \mathbb{N})$ of nonempty subsets of X has $\bigcap_{n\in\mathbb{N}} \overline{F_n} \neq \varnothing$, provided that

(i) each $n \in \mathbb{N}$, there exists an $A_n \in \mathscr{A}_n$ such that $F_n \subseteq A_n$ and

(ii) $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$.

Clearly all metric spaces and all countably Čech-complete spaces are pointwise countably complete. In the other direction, all pointwise countably complete spaces are q-spaces.

Exercise 1.6.17. *Let (Y,τ') be a topological space. Show that both $(f,g) \mapsto f \vee g$ and $(f,g) \mapsto f \wedge g$ are continuous functions, acting from $C_p(Y) \times C_p(Y)$ into $C_p(Y)$.*

Hint: For each $f, g \in C(Y)$ and $x \in Y$,

(i) $(f \vee g)(x) = \max\{f(x), g(x)\} = \frac{1}{2}[f(x) + g(x)] + \frac{1}{2}|f(x) - g(x)|$ *and*

(ii) $(f \wedge g)(x) = \min\{f(x), g(x)\} = \frac{1}{2}[f(x) + g(x)] - \frac{1}{2}|f(x) - g(x)|$.

Therefore, if $f, f'g, g' \in C(Y)$ and $x \in Y$ then, by part (i)

$$|(f \vee g)(x) - (f' \vee g')(x)| \leq |(f - f')(x)| + |(g - g')(x)|$$

and by part (ii)

$$|(f \wedge g)(x) - (f' \wedge g')(x)| \leq |(f - f')(x)| + |(g - g')(x)|.$$

For our considerations, the most important result concerning the lattice structure of $C(Y)$-spaces, where (Y, τ') is a compact space, is the following "lattice formulation" of the famous Stone-Weierstrass Theorem, [47]. Recall that a subset L of $C(Y)$ is called a *sub-lattice* if it is closed under the operations of $\vee$ and $\wedge$. That is, for every $f, g \in C(Y)$, if $f, g \in L$ then $f \vee g \in L$ and $f \wedge g \in L$.

Theorem 1.6.18 (Stone-Weierstrass Theorem, [47])**.** *Let (Y, τ') be a compact topological space, $f \in C(Y)$, $0 < \varepsilon$ and L a sub-lattice of $C(Y)$. If, for each pair of points $x, y \in Y$, there exists an $l_{(x,y)} \in L$ such that $|f(x) - l_{(x,y)}(x)| < \varepsilon$ and $|f(y) - l_{(x,y)}(y)| < \varepsilon$, then there exists a function $l \in L$ such that $\|f - l\|_\infty < \varepsilon$.*

Proof. Let $f \in C(Y)$ and $0 < \varepsilon$ be given. Fix $x \in Y$. For each $y \in Y$ there exists an open neighbourhood U_y^x of y and an element $l_{(x,y)} \in L$ such that

$$l_{(x,y)}(x) < f(x) + \varepsilon \quad \text{and} \quad f(u) - \varepsilon < l_{(x,y)}(u) \quad \text{for all } u \in U_y^x.$$

Let $\{U_{y_j}^x : 1 \le j \le n\}$ be a finite subcover of $\{U_y^x : y \in Y\}$ and let $l_x : Y \to \mathbb{R}$ be defined by,

$$l_x(z) := \max_{1 \le j \le n} l_{(x,y_j)}(z) \quad \text{for all } z \in Y$$

i.e., $l_x = \bigvee_{1 \le j \le n} l_{(x,y_j)} \in L$. Then $l_x(x) < f(x) + \varepsilon$ while, $f(z) - \varepsilon < l_x(z)$ for all $z \in Y$.

We now consider the family of functions $\{l_x : x \in Y\}$. For each $x \in Y$ there exists an open neighbourhood V_x of x such that $l_x(v) < f(v) + \varepsilon$ for all $v \in V_x$. Let $\{V_{x_j} : 1 \le j \le m\}$ be a finite subcover of $\{V_x : x \in Y\}$ and define $l : T \to \mathbb{R}$ by,

$$l(z) := \min_{1 \le j \le m} l_{x_j}(z) \quad \text{for all } z \in Y$$

i.e., $l = \bigwedge_{1 \le j \le m} l_{x_j} \in L$. It is easily seen that $|f(z) - l(z)| < \varepsilon$ for each $z \in Y$ and so $\|f - l\|_\infty < \varepsilon$. ☐

Exercise 1.6.19. *Let (Y, τ') be a compact topological space and let L be a sub-lattice of $C(Y)$. Show that $\overline{L}^{\tau_p(Y)} = \overline{L}^{\|\cdot\|_\infty}$. Hint: It is sufficient to show that $\overline{L}^{\tau_p(Y)} \subseteq \overline{L}^{\|\cdot\|_\infty}$.*

Proposition 1.6.20. *Let (X, τ) be a topological space. If $\{C_k : 1 \le k \le n\}$ is a family of closed subsets of (X, τ) and U is a nonempty open subset of X such that $U \subseteq \bigcup_{k=1}^n C_k$ then there exists a $k_0 \in \{1, 2, \ldots, n\}$ and a nonempty open subset W of U such that $W \subseteq C_{k_0}$.*

Proof. Let $m := \min\{|F| : F \subseteq \{1, 2, \ldots, n\}$ and $U \subseteq \bigcup_{k\in F}^n C_k\}$. Then $1 \le m \le n$. If $m = 1$, then $U \subseteq C_{k_0}$ for some $k_0 \in \{1, 2, \ldots, n\}$ and the result is proven with $W := U$. So let us consider the case when $2 \le m$. In this case we choose $F \subseteq \{1, 2, \ldots, n\}$ such that $|F| = m$ and $U \subseteq \bigcup_{k\in F} C_k$. Then we choose any $k_0 \in F$ and note that $U \not\subseteq \bigcup_{k \in F\setminus\{k_0\}} C_k$. Let $W := U \setminus \bigcup_{k\in F\setminus\{k_0\}} C_k$. Then W is a nonempty open subset of U and $W \subseteq C_{k_0}$. ☐

Corollary 1.6.21. *Let (X, τ) and (Y, τ') be topological spaces and let $f : X \to Y$ be a continuous function. If $\{C_k : 1 \le k \le n\}$ is a family of closed subsets of Y and U is a nonempty open subset of X such that $f(U) \subseteq \bigcup_{k=1}^n C_k$ then there exists a $k_0 \in \{1, 2, \ldots, n\}$ and a nonempty open subset W of U such that $f(W) \subseteq C_{k_0}$.*

Proof. For each $k \in \{1, 2, \ldots, n\}$, let $U_k := \{u \in U : f(u) \in C_k\}$. Then $\{U_k : 1 \leq k \leq n\}$ is a closed cover of U. Hence, by Proposition 1.6.20, there is a $k_0 \in \{1, 2, \ldots, n\}$ and nonempty open subset W of U such that $W \subseteq U_{k_0}$. This completes the proof. $\square$

Exercise 1.6.22. *Let Γ be an uncountable set. For each countable subset C of Γ and $f \in \Gamma^\Gamma$ let*

$$N(f, C) := \{g \in \Gamma^\Gamma : g|_C = f|_C\}.$$

Show that:

(i) $\{N(f, C) : f \in \Gamma^\Gamma$ *and C is a countable subset of* $\Gamma\}$ *is a base for a topology on* Γ^Γ. *This topology is called the topology of coincidence on countable sets and denoted* τ_{count}*;*

(ii) $N(f, C)$ *is both open and closed in the topology of coincidence on countable sets;*

(iii) $(\Gamma^\Gamma, \tau_{count})$ *is a Baire space, see Example 1.1.2.*

We shall say that a topological space (X, τ) is *fragmentable* if there exists a metric ρ on X such that for every nonempty subset A of X and every $0 < \varepsilon$ there exists a τ-open subset W of X such that (i) $\varnothing \neq A \cap W$ and (ii) $\rho - \text{diam}(A \cap W) < \varepsilon$, [29]. Note that in general the metric ρ will not generate the topology τ on X. For example, every scattered space is fragmentable (by the discrete metric), but not every scattered space is discrete. Just think of $X := \{0\} \cup \{1/n : n \in \mathbb{N}\}$ with the topology inherited from $\mathbb{R}$.

Theorem 1.6.23 ([32, Theorem 5.1])**.** *Suppose that (X, τ) and (Y, τ') are topological spaces and $f : X \to Y$ is a quasicontinuous function. If (Y, τ') is fragmented by a metric d whose topology on Y is at least as strong as τ' then, f is τ'-continuous at the points of a residual subset of X.*

Proof. Suppose that $f : X \to Y$ is a quasicontinuous function. Let d be a metric on Y, whose topology on Y, is at least as strong as τ' and which possesses the property that: for every nonempty subset A of Y and every $0 < \varepsilon$ there exists an open subset W of Y such that $\varnothing \neq A \cap W$ and $d - \text{diam}(A \cap W) < \varepsilon$. Fix $0 < \varepsilon$ and define

$$O_\varepsilon := \bigcup\{U \in 2^X : \ U \text{ is open and } d - \text{diam}[f(U)] < \varepsilon\}.$$

Clearly, O_ε is open, as it is a union of open sets. We will now show that O_ε is dense in X. To this end, let V be a nonempty open subset of (X, τ). Now, $f(V)$ is a nonempty subset of Y, therefore there exists an open subset W of Y such that $\varnothing \neq f(V) \cap W$ and $d - \text{diam}[f(V) \cap W] < \varepsilon$. By the definition of quasicontinuity, there exists a nonempty open subset U of V such that $f(U) \subseteq W$. Thus, $f(U) \subseteq f(V) \cap W$ and so $d - \text{diam}[f(U)] < \varepsilon$.

Therefore, $\varnothing \neq U \subseteq V \cap O_\varepsilon$. It now only remains to check that f is d-continuous, and hence τ'-continuous, at each point of $\bigcap_{n \in \mathbb{N}} O_{1/n}$. $\square$

Bibliography

[1] L. Alaoglu, *Weak topologies of normed linear spaces*, Ann. of Math. (2) **41** (1940), 252–267.

[2] A. V. Arhangel'skiĭ, *On topological spaces which are complete in the sense of Čech*, Vestnik Moskov. Univ. Ser. I Mat. Meh. **1961** (1961), no. 2, 37–40. MR 0131258

[3] ———, *Topological function spaces*, Mathematics and its Applications (Soviet Series), vol. 78, Kluwer Academic Publishers Group, Dordrecht, 1992, Translated from the Russian by R. A. M. Hoksbergen. MR 1144519

[4] J. Bourgain, *l^∞/c_0 has no equivalent strictly convex norm*, Proc. Amer. Math. Soc. **78** (1980), no. 2, 225–226. MR 550499

[5] Ahmed Bouziad, *Jeux topologiques et points de continuité d'une application séparément continue*, C. R. Acad. Sci. Paris Sér. I Math. **310** (1990), no. 6, 359–361. MR 1046512

[6] Jiling Cao and Warren B. Moors, *A survey on topological games and their applications in analysis*, RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **100** (2006), no. 1-2, 39–49. MR 2267399

[7] B. Cascales, I. Namioka, and G. Vera, *The Lindelöf property and fragmentability*, Proc. Amer. Math. Soc. **128** (2000), no. 11, 3301–3309. MR 1695167

[8] Gustave Choquet, *Lectures on analysis. Vol. I: Integration and topological vector spaces*, Edited by J. Marsden, T. Lance and S. Gelbart, W. A. Benjamin, Inc., New York-Amsterdam, 1969. MR 0250011

[9] Gabriel Debs, *Points de continuité d'une fonction séparément continue*, Proc. Amer. Math. Soc. **97** (1986), no. 1, 167–176. MR 831408

[10] Robert Deville and Gilles Godefroy, *Some applications of projective resolutions of identity*, Proc. London Math. Soc. (3) **67** (1993), no. 1, 183–199. MR 1218125

[11] N. Dunford and J. T. Schwartz, *Linear Operators. I. General Theory*, Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London, 1958.

[12] R. Engelking, *General topology*, Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989.

[13] Marián J. Fabian, *Gâteaux differentiability of convex functions and topology*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, Inc., New York, 1997, Weak Asplund spaces, A Wiley-Interscience Publication. MR 1461271

[14] W. G. Fleissner and K. Kunen, *Barely Baire spaces*, Fund. Math. **101** (1978), no. 3, 229–240. MR 521125

[15] Zdeněk Frolík, *Generalizations of the G_δ-property of complete metric spaces*, Czechoslovak Math. J **10 (85)** (1960), 359–379. MR 0116305

[16] J. R. Giles, P. S. Kenderov, W. B. Moors, and S. D. Sciffer, *Generic differentiability of convex functions on the dual of a Banach space*, Pacific J. Math. **172** (1996), no. 2, 413–431. MR 1386625

[17] Gary Gruenhage, *Infinite games and generalizations of first-countable spaces*, General Topology and Appl. **6** (1976), no. 3, 339–352. MR 0413049

[18] S. P. Gul'ko, *The structure of spaces of continuous functions and their hereditary paracompactness*, Uspekhi Mat. Nauk **34** (1979), no. 6(210), 33–40. MR 562814

[19] Richard Haydon, *Countable unions of compact spaces with the Namioka property*, Mathematika **41** (1994), no. 1, 141–144. MR 1288058

[20] ———, *Baire trees, bad norms and the Namioka property*, Mathematika **42** (1995), no. 1, 30–42. MR 1346669

[21] ———, *Locally uniformly convex norms in Banach spaces and their duals*, J. Funct. Anal. **254** (2008), no. 8, 2023–2039. MR 2402082

[22] J. E. Jayne, I. Namioka, and C. A. Rogers, *σ-fragmentable Banach spaces*, Mathematika **39** (1992), no. 1, 161–188. MR 1176478

[23] ———, *σ-fragmentable Banach spaces*, Mathematika **39** (1992), no. 2, 197–215. MR 1203276

[24] ———, *Fragmentability and σ-fragmentability*, Fund. Math. **143** (1993), no. 3, 207–220. MR 1247801

[25] ———, *Topological properties of Banach spaces*, Proc. London Math. Soc. (3) **66** (1993), no. 3, 651–672. MR 1207552

[26] ———, *σ-fragmented Banach spaces. II*, Studia Math. **111** (1994), no. 1, 69–80. MR 1292853

[27] ———, *Continuous functions on compact totally ordered spaces*, J. Funct. Anal. **134** (1995), no. 2, 261–280. MR 1363800

[28] ———, *Continuous functions on products of compact Hausdorff spaces*, Mathematika **46** (1999), no. 2, 323–330. MR 1832623

[29] J. E. Jayne and C. A. Rogers, *Borel selectors for upper semicontinuous set-valued maps*, Acta Math. **155** (1985), no. 1-2, 41–79. MR 793237

[30] P. S. Kenderov, I. S. Kortezov, and W. B. Moors, *Continuity points of quasi-continuous mappings*, Topology Appl. **109** (2001), no. 3, 321–346. MR 1807395

[31] ———, *Norm continuity of weakly continuous mappings into Banach spaces*, Topology Appl. **153** (2006), no. 14, 2745–2759. MR 2243747

[32] Petar S. Kenderov and Warren B. Moors, *Fragmentability and sigma-fragmentability of Banach spaces*, J. London Math. Soc. (2) **60** (1999), no. 1, 203–223. MR 1721825

[33] ———, *Separate continuity, joint continuity and the Lindelöf property*, Proc. Amer. Math. Soc. **134** (2006), no. 5, 1503–1512. MR 2199199

[34] ———, *Fragmentability of groups and metric-valued function spaces*, Topology Appl. **159** (2012), no. 1, 183–193. MR 2852961

[35] M. R. Krom, *Cartesian products of metric Baire spaces*, Proc. Amer. Math. Soc. **42** (1974), 588–594. MR 0334138

[36] Aníbal Moltó, José Orihuela, Stanimir Troyanski, and Manuel Valdivia, *A nonlinear transfer technique for renorming*, Lecture Notes in Mathematics, vol. 1951, Springer-Verlag, Berlin, 2009. MR 2462399

[37] W. B. Moors and J. R. Giles, *Generic continuity of minimal set-valued mappings*, J. Austral. Math. Soc. Ser. A **63** (1997), no. 2, 238–262. MR 1475564

[38] I. Namioka and R. Pol, *Mappings of Baire spaces into function spaces and Kadec renorming*, Israel J. Math. **78** (1992), no. 1, 1–20. MR 1194955

[39] John Nash, *Non-cooperative games*, Ann. of Math. (2) **54** (1951), 286–295. MR 0043432

[40] John F. Nash, Jr., *Equilibrium points in n-person games*, Proc. Nat. Acad. Sci. U. S. A. **36** (1950), 48–49. MR 0031701

[41] John C. Oxtoby, *The Banach-Mazur game and Banach category theorem*, Contributions to the theory of games, vol. 3, Annals of Mathematics Studies, no. 39, Princeton University Press, Princeton, N. J., 1957, pp. 159–163. MR 0093741

[42] ———, *Measure and category. A survey of the analogies between topological and measure spaces*, Springer-Verlag, New York-Berlin, 1971, Graduate Texts in Mathematics, Vol. 2. MR 0393403

[43] N. K. Ribarska, *Internal characterization of fragmentable spaces*, Mathematika **34** (1987), no. 2, 243–257. MR 933503

[44] V. I. Rybakov, *Yet another class of Namioka spaces*, Mat. Zametki **73** (2003), no. 2, 263–268. MR 1997666

[45] Jean Saint-Raymond, *Jeux topologiques et espaces de Namioka*, Proc. Amer. Math. Soc. **87** (1983), no. 3, 499–504. MR 684646

[46] G. A. Sokolov, *On some classes of compact spaces lying in Σ-products*, Comment. Math. Univ. Carolin. **25** (1984), no. 2, 219–231. MR 768809

[47] M. H. Stone, *The generalized Weierstrass approximation theorem*, Math. Mag. **21** (1948), 167–184, 237–254. MR 0027121

[48] Michel Talagrand, *Espaces de Banach faiblement $\mathcal{K}$-analytiques*, Ann. of Math. (2) **110** (1979), no. 3, 407–438. MR 554378

[49] ———, *Espaces de Baire et espaces de Namioka*, Math. Ann. **270** (1985), no. 2, 159–164. MR 771977

[50] Rastislav Telgársky, *Topological games: on the 50th anniversary of the Banach-Mazur game*, Rocky Mountain J. Math. **17** (1987), no. 2, 227–276. MR 892457

[51] S. Todorčević, *Trees and linearly ordered sets*, Handbook of set-theoretic topology, North-Holland, Amsterdam, 1984, pp. 235–293. MR 776625

[52] J. v. Neumann, *Zur Theorie der Gesellschaftsspiele*, Math. Ann. **100** (1928), no. 1, 295–320. MR 1512486

[53] L. Vašák, *On one generalization of weakly compactly generated Banach spaces*, Studia Math. **70** (1981), no. 1, 11–19. MR 646957

[54] John von Neumann and Oskar Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, New Jersey, 1944. MR 0011937

Index of notation

- Suppose that X and Y are sets and $\Phi : X \to 2^Y$ is a set-valued mapping. Then
  - $\Phi(U) := \bigcup\{\Phi(u) : u \in U\}$;
  - $\mathrm{Dom}(\Phi) := \{x \in X : \Phi(x) \neq \varnothing\}$. $\mathrm{Dom}(\Phi)$ is called the *domain of* Φ or the *effective domain of* Φ;
  - If X posseses a topology τ then we say that Φ is *densely defined* if $\mathrm{Dom}(\Phi)$ is dense in (X, τ);
  - If A is a subset of X and $\Phi : X \to 2^Y$ then $\Phi|_A : A \to 2^Y$ is defined by, $\Phi|_A(a) := \Phi(a)$ for all $a \in A$;
  - $\mathrm{Gr}(\Phi) := \{(x, y) \in X \times Y : y \in \Phi(x)\}$, called the *graph of* Φ.
- If (M, d) is a metric space then
  - If $0 < r < \infty$ and $x \in M$ then $B[x, r] := \{y \in M : d(x, y) \leq r\}$ is call the *closed ball of radius* r *with centre* x;
  - If $0 < r < \infty$ and $x \in M$ then $B(x, r) := \{y \in M : d(x, y) \leq r\}$ is call the *open ball of radius* r *with centre* x;
  - If $0 < r < \infty$ and $x \in M$ then $S(x, r) := \{y \in M : d(x, y) = r\}$ is call the *sphere of radius* r *with centre* x;
  - If $0 < r < \infty$ and A is a subset of M then $B(A; r) := \bigcup_{a \in A} B(a; r)$ and $B[A; r] := \bigcup_{a \in A} B[a; r]$;
  - $S_\varepsilon(M) := \{X \in 2^M :$ for every distinct $x, y \in X$, $\varepsilon \leq d(x, y)\}$, the members of $S_\varepsilon(M)$ are called *ε-separated sets* or else, *ε-nets*;
  - For a subset C of M and an element $x \in M$ we define, the *distance from* x *to* C by, $d(x, C) := \inf\{d(x, c) : c \in C\}$;
  - $\mathcal{B}^1(X, M)$ denotes the set of all *Baire one* functions from a topological space (X, τ) into M;
  - $\mathcal{B}^1(X)$ denotes the set of all real-valued Baire one functions from a topological space (X, τ).
- The *natural numbers*, $\mathbb{N} := \{1, 2, 3, \ldots\}$.
- The *integers*, $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2 \ldots\}$.

- The *rational numbers*, $\mathbb{Q} := \{a/b : a, b \in \mathbb{Z}, b \neq 0\}$.
- The *real numbers*, $\mathbb{R}$.
- For any set X, 2^X is the set of all subsets of X (sometimes we use $\mathcal{P}(X)$ to denote the set of all subsets of X as well) .
- For any subset A of a topological space (X, τ), we define
  - $\text{int}(A)$, called the *interior* of A, is the union of all open sets contained in A;
  - $\overline{A}$, called the *closure* of A, is the intersection of all closed sets containing A;
  - $\text{Bd}(A)$, called the *boundary* of A, is $\overline{A} \setminus \text{int}(A)$,
- For any points x and y in a vector space X, we define the following intervals:
  - $[x, y] := \{x + \lambda(y - x) : 0 \leq \lambda \leq 1\}$;
  - $(x, y) := \{x + \lambda(y - x) : 0 < \lambda < 1\}$;
  - $[x, y) := \{x + \lambda(y - x) : 0 \leq \lambda < 1\}$;
  - $(x, y] := \{x + \lambda(y - x) : 0 < \lambda \leq 1\}$.
- For any normed linear space $(X, \|\cdot\|)$, we define
  - $B[x, r] := \{y \in X : \|x - y\| \leq r\}$, for any $x \in X$ and $r > 0$;
  - $B(x; r) := \{y \in X : \|x - y\| < r\}$, for any $x \in X$ and $r > 0$;
  - $B_X := B[0, 1]$;
  - $S_X := \{x \in X : \|x\| = 1\}$.
- Given a compact Hausdorff space K, we write $C(K)$ for the set of all real-valued continuous functions on K. This is a vector space under the operations of pointwise addition and pointwise scalar multiplication. $C(K)$ becomes a Banach space when equipped with the uniform norm $\|\cdot\|_\infty$, defined by

$$\|f\|_\infty := \sup_{x \in K} |f(x)|, \quad \text{for all } f \in C(K).$$

- For a normed linear space $(X, \|\cdot\|)$, X^*, the set of bounded linear maps from X to $\mathbb{R}$, is called the *dual space* of X. X^* is a Banach space when equipped with the operator norm, given by

$$\|f\| := \sup_{x \in B_X} |f(x)|, \qquad \text{for all } f \in X^*.$$

- Let X be a set and Y a totally ordered set. For any function $f : X \to Y$ we define

$$\text{argmax}(f) := \{x \in X : f(y) \leq f(x) \text{ for all } y \in X\},$$
$$\text{argmin}(f) := \{x \in X : f(x) \leq f(y) \text{ for all } y \in X\}.$$

- Let A be a subset of a vector space X. Then the *convex hull* of A, denoted by $\text{co}(A)$, is defined to be the intersection of all convex subsets of X that contain A.

- Let X be a set and let $f : X \to \mathbb{R} \cup \{\infty\}$ a function. Then

$$\text{Dom}(f) := \{x \in X : f(x) < \infty\}.$$

  We say that the function f is a *proper function* if $\text{Dom}(f) \neq \varnothing$.

- Let $(X, \|\cdot\|)$ be a normed linear space and $f : X \to [-\infty, \infty]$. Then the *Fenchel conjugate* of f is the function $f^* : X^* \to [-\infty, \infty]$ defined by,

$$f^*(x^*) := \sup_{x \in X}\{x^*(x) - f(x)\}.$$

  The function f^* is convex and if f is a proper function then f^* never takes the value $-\infty$.

- If f is a convex function defined on a nonempty convex subset K of a normed linear space $(X, \|\cdot\|)$ and $x \in K$, then we define the *subdifferential of f at x* to be the set $\boldsymbol{\partial f(x)}$ of all $x^* \in X^*$ satisfying

$$x^*(y - x) \leq f(y) - f(x) \quad \text{for all } y \in K.$$

- It is assumed that the reader has a basic working knowledge of metric spaces, normed linear spaces and even basic general topology. In particular, it is assumed that the reader is familiar with Tychonoff's theorem, the Banach-Alaoglu theorem and the Separation theorem.

  Theorem (Tychonoff's Theorem [12]). *The Cartesian product $\prod_{s \in S} X_s$, where $X_s \neq \varnothing$ for all $s \in S$, is compact if, and only if, all spaces X_s are compact.*

  Theorem (Banach-Alaoglu Theorem [1]). *Let $(X, \|\cdot\|)$ be a normed linear space. Then $(B_{X^*}, weak^*)$ is compact.*

  Theorem (Separation Theorem [11, page 418]). *Suppose that (X, τ) is a locally convex space over $\mathbb{R}$ and C is a nonempty closed convex subset of X. If $x_0 \notin C$ then there exists a continuous linear functional x^* on X such that*

$$\sup\{x^*(c) : c \in C\} < x^*(x_0).$$

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