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Topological games and topological groups [☆]

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Abstract

A semitopological group (topological group) is a group endowed with a topology for which multiplication is separately continuous (multiplication is jointly continuous and inversion is continuous). In this paper we give some topological conditions on a semitopological group that imply that it is a topological group. In particular, we show that every almost Čech-complete semitopological group is a topological group. Thus we improve some recent results of A. Bouziad. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The purpose of this note is to refine and improve upon some of the recent advances made by Bouziad on the question of when a semitopological group is in fact a topological group. Recall that a *semitopological group* (*paratopological group*) is a group endowed with a topology for which multiplication is separately (jointly) continuous. Research on this question possibly began in [13] when Montgomery showed that each completely metrizable semitopological group is a paratopological group. Later in 1957 Ellis showed that each locally compact semitopological group is in fact a topological group (see [5,6]). This answered a question raised by Wallace in [18]. Then in 1960 Zelazko used

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Montgomery's result from [13] to show that each completely metrizable semitopological group is a topological group. Much later in [2] Bouziad improved both of these results and answered a question raised by Pfister in [16] by showing that each Čech-complete semitopological group is a topological group. (Recall that both locally compact and completely metrizable topological spaces are Čech-complete.) To do this, it was sufficient for Bouziad to show that each Čech-complete semitopological group is a paratopological group since earlier, Brand (see [4]) had proven that every Čech-complete paratopological group is a topological group. Brand's proof of this was later improved and simplified in [16]. Let us also mention here, that apart from the a fore-mentioned authors there have been many other important contributions to the problem of determining when a semitopological group is a topological group, including, [12,7,14,8] to name but a few. Our contribution to this problem is based upon the following game. Let (X,τ) be a topological space and let D be a dense subset of X. On X we consider the $\mathcal{G}_S(D)$*-game* played between two players α and β. Player β goes first (always!) and chooses a non-empty open subset $B_1 \subseteq X$. Player α must then respond by choosing a non-empty open subset $A_1 \subseteq B_1$. Following this, player β must select another non-empty open subset $B_2 \subseteq A_1 \subseteq B_1$ and in turn player α must again respond by selecting a non-empty open subset $A_2 \subseteq B_2 \subseteq A_1 \subseteq B_1$. Continuing this procedure indefinitely the players α and β produce a sequence $((A_n, B_n)\colon n \in \mathbb{N})$ of pairs of open sets called a *play* of the $\mathcal{G}_S(D)$-game. We shall declare that α *wins* a play $((A_n, B_n)\colon n \in \mathbb{N})$ of the $\mathcal{G}_S(D)$-game if; $\bigcap_{n\in\mathbb{N}} A_n$ is non-empty and each sequence $(a_n\colon n \in \mathbb{N})$ with $a_n \in A_n \cap D$ has a cluster-point in X. Otherwise the player β is said to have won this play. By a *strategy* t for the player β we mean a '*rule*' that specifies each move of the player β in every possible situation. More precisely, a strategy $t := (t_n\colon n \in \mathbb{N})$ for β is a sequence of τ-valued functions such that $t_{n+1}(A_1, \dots, A_n) \subseteq A_n$ for each $n \in \mathbb{N}$. The domain of each function t_n is precisely the set of all finite sequences $(A_1, A_2, \dots, A_{n-1})$ of length $n-1$ in τ with $A_j \subseteq t_j(A_1, \dots, A_{j-1})$ for all $1 \leqslant j \leqslant n-1$. (Note: the sequence of length 0 will be denoted by $\emptyset$.) Such a finite sequence $(A_1, A_2, \dots, A_{n-1})$ or infinite sequence $(A_n\colon n \in \mathbb{N})$ is called a t*-sequence*. A strategy $t := (t_n\colon n \in \mathbb{N})$ for the player β is called a *winning strategy* if each t-sequence is won by β. We will call a topological space (X,τ) a *strongly Baire* or (*strongly* β*-unfavorable*) space if it is regular and there exists a dense subset D of X such that the player β does **not** have a winning strategy in the $\mathcal{G}_S(D)$-game played on X. It follows from Theorem 1 in [17] that each strongly Baire space is in fact a Baire space and it is easy to see that each strongly Baire space has at least one q_D-point. Indeed, if $t := (t_n\colon n \in \mathbb{N})$ is any strategy for β then there is a t-sequence $(A_n\colon n \in \mathbb{N})$ where α wins. In this case we have that each point of $\bigcap_{n\in\mathbb{N}} A_n$ is a q_D-point. Recall that a point $x \in X$ is called a q_D-point (with respect to some dense subset D of X) if there exists a sequence of neighborhoods $(U_n\colon n \in \mathbb{N})$ of x such that every sequence $(x_n\colon n \in \mathbb{N})$ with $x_n \in U_n \cap D$ has a cluster-point in X.

The remainder of this paper is divided into 3 parts. In the next section we will show that every strongly Baire semitopological group is a paratopological group and then in Section 3 we will show that each strongly Baire semitopological group is in fact a topological group. Finally, in Section 4 we will provide some examples of topological spaces that are strongly Baire.

2. Paratopological groups

We begin with some definitions. Let X, Y and Z be topological spaces, then we will say that a function $f:X \times Y \to Z$ is *strongly quasi-continuous at* $(x, y) \in X \times Y$ if for each neighborhood W of $f(x, y)$ and each product of open sets $U \times V \subseteq X \times Y$ containing (x, y) there exists a non-empty open subset $U' \subseteq U$ and a neighborhood V' of y such that $f(U' \times V') \subseteq W$ [3]. If f is strongly quasi-continuous at each point $(x, y) \in X \times Y$ then we say that f is *strongly quasi-continuous on* $X \times Y$. Finally, a function $f:X \times Y \to Z$ is said to be *separately continuous* on $X \times Y$ if for each $x_0 \in X$ and $y_0 \in Y$ the functions $y \to f(x_0, y)$ and $x \to f(x, y_0)$ are both continuous on Y and X, respectively. Note: in the following results 'e' will denote the identity element of the group $(G, \cdot)$.

Lemma 1 (Theorem 1 [3]). *Let X be a strongly Baire space, Y a topological space and Z a regular space. If $f:X \times Y \to Z$ is a separately continuous function and D is a dense subset of Y then for each q_D-point $y_0 \in Y$ the mapping f is strongly quasi-continuous at each point of $X \times \{y_0\}$.*

Proof. Let D_X be any dense subset of X such that β does not have a winning strategy in the $\mathcal{G}_S(D_X)$-game played on X. (Note: such a dense subset is guaranteed by the fact that X is a strongly Baire space.) We need to show that f is strongly quasi-continuous at each point $(x_0, y_0) \in X \times \{y_0\}$. So in order to obtain a contradiction let us assume that f is not quasi-continuous at some point $(x_0, y_0) \in X \times \{y_0\}$. Then, by the regularity of Z there exist open neighborhoods W of $f(x_0, y_0)$, U of x_0 and V of y_0 so that $f(U' \times V') \not\subseteq \overline{W}$ for each non-empty open subset U' of U and neighborhood $V' \subseteq V$ of y_0. Again, by the regularity of Z there exists an open neighborhood W' of $f(x_0, y_0)$ so that $\overline{W'} \subseteq W$. Note that by possibly making U smaller we may assume that $f(x, y_0) \in W'$ for all $x \in U$. We will now inductively define a strategy $t := (t_n\colon n \in \mathbb{N})$ for the player β in the $\mathcal{G}_S(D_X)$-game played on X, but first we shall denote by $(O_n\colon n \in \mathbb{N})$ any sequence of open neighborhoods of y_0 with the property that each sequence $(y_n\colon n \in \mathbb{N})$ in D with $y_n \in O_n$ has a cluster-point in Y.

Step 1. Let $V_1 := \{y \in V \cap O_1\colon f(x_0, y) \in W'\}$ and choose $(x_1, y_1) \in (U \cap D_X) \times (V_1 \cap D)$ so that $f(x_1, y_1) \notin \overline{W}$. Then define, $t_1(\emptyset) := \{x \in U\colon f(x, y_1) \notin \overline{W}\}$.

Now suppose that (x_j, y_j), V_j and t_j have been defined for each t-sequence $(A_1, A_2, \ldots, A_{j-1})$ of length $(j-1)$, $1 \leqslant j \leqslant n$ so that,

(i) $y_0 \in V_j := \{y \in V_{j-1} \cap O_j\colon f(x_{j-1}, y) \in W'\}$;

(ii) $f(x_j, y_j) \notin \overline{W}$ and $(x_j, y_j) \in (A_{j-1} \cap D_X) \times (V_j \cap D)$;

(iii) $t_j(A_1, \ldots, A_{j-1}) := \{x \in A_{j-1}\colon f(x, y_j) \notin \overline{W}\}$.

Step $n+1$. For each t-sequence $(A_1, \ldots, A_n)$ of length n we select $(x_{n+1}, y_{n+1}) \in X \times Y$ and open sets V_{n+1} and $t_{n+1}(A_1, \ldots, A_n)$ so that,

(i) $y_0 \in V_{n+1} := \{y \in V_n \cap O_{n+1}\colon f(x_n, y) \in W'\}$;

(ii) $f(x_{n+1}, y_{n+1}) \notin \overline{W}$ and $(x_{n+1}, y_{n+1}) \in (A_n \cap D_X) \times (V_{n+1} \cap D)$;

(iii) $t_{n+1}(A_1, \ldots, A_n) := \{x \in A_n\colon f(x, y_{n+1}) \notin \overline{W}\}$.

This completes the definition of $t := (t_n\colon\ n \in \mathbb{N})$. Now since t is not a winning strategy for the player β in the $\mathcal{G}_S(D_X)$-game there exists a t-sequence $(A_n\colon\ n \in \mathbb{N})$ where α wins and since $x_{n+1} \in A_n \cap D_X$ and $y_n \in O_n \cap D$ for each $n \in \mathbb{N}$ both sequences $(x_n\colon\ n \in \mathbb{N})$ and $(y_n\colon\ n \in \mathbb{N})$ have cluster-points. Let x_∞ be any cluster-point of $(x_n\colon\ n \in \mathbb{N})$ and y_∞ be any cluster-point of $(y_n\colon\ n \in \mathbb{N})$. Then for each fixed $n \in \mathbb{N}$, $f(x_n, y_k) \in f(\{x_n\} \times V_k) \subseteq f(\{x_n\} \times V_{n+1}) \subseteq W'$ for all $n < k$ since $y_k \in V_k \subseteq V_{n+1}$. Therefore $f(x_n, y_\infty) \in \overline{W'}$ for each $n \in \mathbb{N}$ and so $f(x_\infty, y_\infty) \in \overline{W'} \subseteq W$. On the other hand if we again fix $n \in \mathbb{N}$ then $f(x_{k+1}, y_n) \in f(t_k(A_1, \ldots, A_{k-1}) \times \{y_n\}) \subseteq f(t_n(A_1, \ldots, A_{n-1}) \times \{y_n\}) \subseteq X \setminus \overline{W}$ for all $n \leqslant k$ since $x_{k+1} \in A_k \subseteq t_k(A_1, \ldots, A_{k-1}) \subseteq t_n(A_1, \ldots, A_{n-1})$. Therefore, $f(x_\infty, y_n) \notin W$ for each $n \in \mathbb{N}$ and so $f(x_\infty, y_\infty) \notin W$. This however, contradicts our earlier conclusion that $f(x_\infty, y_\infty) \in W$. Hence f is strongly quasi-continuous at (x_0, y_0). □

The origins of the following lemma may be traced back to Theorem 3 in [3].

Lemma 2. *Let $(G, \cdot, \tau)$ be a regular semitopological group. If there exists a dense subset D of G and a sequence of neighborhoods $(U_n\colon\ n \in \mathbb{N})$ of e so that every sequence $(z_n\colon\ n \in \mathbb{N})$ in D with $z_n \in U_n \cdot U_n$ has a cluster-point in G and multiplication is strongly quasi-continuous at (e, e) then $(G, \cdot, \tau)$ is a paratopological group.*

Proof. Since $(G, \cdot, \tau)$ is a semitopological group it is sufficient to show that the (separately continuous and open) mapping $\pi : G \times G \to G$ defined by, $\pi(g, h) := g \cdot h$ is jointly continuous at (e, e). So in order to obtain a contradiction we will assume that π is not jointly continuous at (e, e). Therefore by the regularity of (G, τ) there exists an open neighborhood W of e so that for every neighborhood U of e, $U \cdot U \not\subseteq \overline{W}$. Again, by the regularity of (G, τ) there exists an open neighborhood V of e so that $\overline{V} \subseteq W$. Let $V^* := \{g \in G\colon\ (g, e) \in \operatorname{int} \pi^{-1}(V)\}$. Then by the strong quasi-continuity of π at (e, e), $e \in \overline{V^*}$. We will now inductively define sequences $(z_n\colon\ n \in \mathbb{N})$ and $(v_n\colon\ n \in \mathbb{N})$ in D and decreasing neighborhoods $(Z_n\colon\ n \in \mathbb{N})$ and $(V_n\colon\ n \in \mathbb{N})$ of e.

Step 1. Choose $v_1 \in V^* \cap D$ and a neighborhood Z_1 of e so that $Z_1 \subseteq U_1$ and $(v_1 \cdot Z_1) \cdot Z_1 \subseteq V$. Then choose $z_1 \in (Z_1 \cdot Z_1 \setminus \overline{W}) \cap D$ and a neighborhood V_1 of e so that $V_1 \subseteq U_1$ and $V_1 \cdot z_1 \subseteq G \setminus \overline{W}$. For purely notational reasons we will define $V_0 := U_0 := G$. Now suppose that $v_j, z_j \in D$ and Z_j, V_j have been defined for each $1 \leqslant j \leqslant n$ so that,

(i) $v_j \in (V^* \cap V_{j-1}) \cap D$ and $(v_j \cdot Z_j) \cdot Z_j \subseteq V$;

(ii) $z_j \in (Z_j \cdot Z_j \setminus \overline{W}) \cap D$ and $V_j \cdot z_j \subseteq G \setminus \overline{W}$;

(iii) $Z_j \subseteq Z_{j-1} \cap U_j$ and $V_j \subseteq V_{j-1} \cap U_j$.

Step $n+1$. Choose $v_{n+1} \in (V^* \cap V_n) \cap D$ and a neighborhood Z_{n+1} of e so that $Z_{n+1} \subseteq Z_n \cap U_{n+1}$ and $(v_{n+1} \cdot Z_{n+1}) \cdot Z_{n+1} \subseteq V$. Then choose $z_{n+1} \in (Z_{n+1} \cdot Z_{n+1} \setminus \overline{W}) \cap D$ and a neighborhood V_{n+1} of e so that $V_{n+1} \subseteq V_n \cap U_{n+1}$ and $V_{n+1} \cdot z_{n+1} \subseteq G \setminus \overline{W}$. This completes the induction. Now since $z_j \in Z_j \cdot Z_j \subseteq U_j \cdot U_j$ and $v_{j+1} = v_{j+1} \cdot e \in V_j \cdot V_j \subseteq U_j \cdot U_j$ for each $j \in \mathbb{N}$, both sequences $(z_n\colon\ n \in \mathbb{N})$ and $(v_n\colon\ n \in \mathbb{N})$ have cluster-points in G. Let z_∞ be any cluster-point of $(z_n\colon\ n \in \mathbb{N})$ and v_∞ be any cluster-point of $(v_n\colon\ n \in \mathbb{N})$. Then for each fixed $n \in \mathbb{N}$, $v_n \cdot z_k \in v_n \cdot Z_k \cdot Z_k \subseteq v_n \cdot Z_n \cdot Z_n \subseteq V$ for all $n \leqslant k$ since $Z_k \subseteq Z_n$. Therefore, $v_n \cdot z_\infty \in \overline{V}$ for each $n \in \mathbb{N}$ and so $v_\infty \cdot z_\infty \in \overline{V} \subseteq W$. On the other hand, if we again fix $n \in \mathbb{N}$ then $v_{k+1} \cdot z_n \in V_k \cdot z_n \subseteq V_n \cdot z_n \subseteq G \backslash \overline{W}$ for all $n \leqslant k$ since $v_{k+1} \in V_k \subseteq V_n$. Therefore, $v_\infty \cdot z_n \in G \backslash W$ for each $n \in \mathbb{N}$ and so $v_\infty \cdot z_\infty \in G \backslash W$. This however, contradicts our earlier conclusion that $v_\infty \cdot z_\infty \in W$. Hence $(G, \cdot, \tau)$ is a paratopological group. □

Theorem 1. *Let $(G, \cdot, \tau)$ be a semitopological group. If (G, τ) is a strongly Baire space then $(G, \cdot, \tau)$ is a paratopological group.*

Proof. Let D_G be any dense subset of G such that β does not have a winning strategy in the $\mathcal{G}_S(D_G)$-game played on G. We begin by observing that e is a q_D-point with respect to some dense subset D of G and so by Lemma 1 the mapping $\pi : G \times G \to G$ defined by $\pi(g, h) := g \cdot h$ is strongly quasi-continuous on $G \times \{e\}$. Hence by Lemma 2 we need only show that there exists a dense subset D^* of G and a sequence of open neighborhoods $(U_n^*\colon n \in \mathbb{N})$ of e so that every sequence $(z_n\colon n \in \mathbb{N})$ in D^* with $z_n \in U_n^* \cdot U_n^*$ has a cluster-point in G. To this end, we will inductively define a strategy $t := (t_n\colon n \in \mathbb{N})$ for β in the $\mathcal{G}_S(D_G)$-game played on G.

Step 1. We define $U_1 := G$, $V_1 := G$ and $t_1(\emptyset) := G$. Now, suppose that U_j, V_j and t_j have been defined for each t-sequence $(A_1, A_2, \ldots, A_{j-1})$ of length $(j-1)$, $1 \leqslant j \leqslant n$ so that,

(i) $U_j \subseteq U_{j-1}$;
(ii) $e \in V_j \subseteq V_{j-1}$;
(iii) $\pi(U_j \times V_j) \subseteq A_{j-1}$;

for each $1 < j \leqslant n$ and $t_j(A_1, \ldots, A_{j-1}) := U_j$ for each $1 \leqslant j \leqslant n$.

Step $n+1$. For each t-sequence $(A_1, \ldots, A_n)$ of length n we choose open sets U_{n+1} and V_{n+1} so that,

(i) $U_{n+1} \subseteq U_n$;
(ii) $e \in V_{n+1} \subseteq V_n$;
(iii) $\pi(U_{n+1} \times V_{n+1}) \subseteq A_n$.

Then we define $t_{n+1}(A_1, \ldots, A_n) := U_{n+1}$. Note: this construction is possible since π is strongly quasi-continuous and $A_n \subseteq t_n(A_1, \ldots, A_{n-1}) = U_n$. This completes the definition of $t := (t_n\colon n \in \mathbb{N})$. Now since t is not a winning strategy for β there exists a t-sequence $(A_n\colon n \in \mathbb{N})$ where α wins and so

$$\bigcap_{n \in \mathbb{N}} A_n = \bigcap_{n \in \mathbb{N}} U_n \text{ is non-empty.}$$

Choose $u \in \bigcap_{n \in \mathbb{N}} A_n$ and set $D^* := u^{-1} D_G$. Then for each $n \in \mathbb{N}$ define, $U_n^* := (u^{-1} \cdot U_n) \cap V_n$. It is now a routine matter to show that every sequence $(z_n\colon n \in \mathbb{N})$ in D^* with $z_n \in U_n^* \cdot U_n^*$ has a cluster-point in G. □

3. Continuity of inversion

Let X and Y be topological spaces. Then a function $f : X \to Y$ is said to be *quasi-continuous at* $x \in X$ if for each neighborhood W of $f(x)$ and neighborhood U of x there exists a non-empty open set $V \subseteq U$ such that $f(V) \subseteq W$ [10]. If f is quasi-continuous at each point $x \in X$ then we say that f is *quasi-continuous on* X. The following result is based upon Theorem 4.2 in [1] which in turn is based upon a clever trick from [16].

Lemma 3. *Let $(G, \cdot, \tau)$ be a semitopological group. If (G, τ) is a strongly Baire space then inversion is quasi-continuous on G.*

Proof. Let us begin with the following preliminary observation. Suppose that U and W are any neighborhoods of e such that $U \cdot U \subseteq W$ and D is any subset of G such that $D^{-1} \subseteq U$. Then $(\overline{D})^{-1} \subseteq W$. To show that inversion is quasi-continuous on G it suffices to show that inversion is quasi-continuous at $e \in G$. So in order to obtain a contradiction let us assume that inversion is not quasi-continuous at $e \in G$. Then there exist neighborhoods U and W of e such that for each non-empty open subset V of U, $V^{-1} \not\subseteq W$. Note that by possibly making U smaller we may assume that $U \cdot U \subseteq W$. Hence for each dense subset D of U and non-empty open subset V of U there exists a point $x \in D \cap V$ such that $x^{-1} \notin U$, for otherwise, $V^{-1} \subseteq (\overline{V \cap D})^{-1} \subseteq W$. Next, we let D be any dense subset of G such that β does not have a winning strategy in the $\mathcal{G}_S(D)$-game played on G and then we apply the above observations to inductively define a strategy $t := (t_n\colon n \in \mathbb{N})$ for β in the $\mathcal{G}_S(D)$-game played on G.

Step 1. We define $x_1 := e$, $U_1 := U$ and $t_1(\emptyset) := x_1 \cdot U_1$. Now, suppose that x_j, U_j and t_j have been defined for each t-sequence $(A_1, \ldots, A_{j-1})$ of length $(j-1)$, $1 \leqslant j \leqslant n$ so that,

(i) $x_j \in (x_1 \cdot x_2 \cdots x_{j-1})^{-1} \cdot (A_{j-1} \cap D)$ and $x_j^{-1} \notin U$;
(ii) $x_1 \cdot x_2 \cdots x_j \cdot U_j \subseteq A_{j-1}$;
(iii) $\overline{U_j} \cdot \overline{U_j} \subseteq U_{j-1}$;

for each $1 < j \leqslant n$ and $t_j(A_1, \ldots, A_{j-1}) := x_1 \cdot x_2 \cdots x_j \cdot U_j$ for each $1 \leqslant j \leqslant n$.

Step $n+1$. For each t-sequence $(A_1, \ldots, A_n)$ of length n we choose an element $x_{n+1} \in G$ and a neighborhood U_{n+1} of e so that,

(i) $x_{n+1} \in (x_1 \cdot x_2 \cdots x_n)^{-1} \cdot (A_n \cap D)$ and $x_{n+1}^{-1} \notin U$;
(ii) $x_1 \cdot x_2 \cdots x_{n+1} \cdot U_{n+1} \subseteq A_n$;
(iii) $\overline{U_{n+1}} \cdot \overline{U_{n+1}} \subseteq U_n$.

Then we define $t_{n+1}(A_1, \ldots, A_n) := x_1 \cdot x_2 \cdots x_{n+1} \cdot U_{n+1}$. Note: this construction is possible since multiplication is jointly continuous (Theorem 1) and $A_n \subseteq x_1 \cdot x_2 \cdots x_n \cdot U_n$. This completes the definition of $t := (t_n\colon n \in \mathbb{N})$. Now since t is not a winning strategy for β there exists a t-sequence $(A_n\colon n \in \mathbb{N})$ where α wins. Hence we see that the sequence $((x_1 \cdot x_2 \cdots x_n)\colon n \in \mathbb{N})$ has a cluster-point $x \in G$. Next we choose $k > n+1$ so that $x_1 \cdot x_2 \cdots x_{k-1} \in x \cdot U_{n+1}$ that is, so that $x_k^{-1} \in (x_1 \cdot x_2 \cdots x_k)^{-1} \cdot x \cdot U_{n+1}$. Now the element $(x_1 \cdot x_2 \cdots x_k)^{-1} \cdot x$ is a cluster-point of the sequence $((x_1 \cdot x_2 \cdots x_k)^{-1} \cdot (x_1 \cdot x_2 \cdots x_{k+j})\colon j \in \mathbb{N})$ and so we have $(x_1 \cdot x_2 \cdots x_k)^{-1} \cdot (x_1 \cdot x_2 \cdots x_{k+j}) = x_{k+1} \cdots x_{k+j} \in U_{k+1} \cdot U_{k+2} \cdots U_{k+j}$. Hence, $(x_1 \cdot x_2 \cdots x_k)^{-1} \cdot x \in \overline{U_k} \subseteq U_{k-1} \subseteq U_{n+1}$. Thus, $x_k^{-1} \in (x_1 \cdot x_2 \cdots x_k)^{-1} \cdot x \cdot U_{n+1} \subseteq U_{n+1} \cdot U_{n+1} \subseteq U_n \subseteq U$; which contradicts the way x_k was chosen. This shows that inversion is quasi-continuous on G. □

Lemma 4. *Let $(G, \cdot, \tau)$ be a paratopological group. If inversion is quasi-continuous at e then $(G, \cdot, \tau)$ is a topological group.*

Proof. Since $(G, \cdot, \tau)$ is a paratopological group it suffices to show that inversion is continuous on G. In fact, because $(G, \cdot, \tau)$ is a semitopological group it will suffice to show that inversion is continuous at $e \in G$. To this end, let W be any neighborhood of e. Since G is a paratopological group there exists a neighborhood U of e so that $U \cdot U \subseteq W$. Now since inversion is quasi-continuous at e there is a non-empty open subset V of U such that $V^{-1} \subseteq U$. Hence, $V \cdot V^{-1}$ is an open neighborhood of e and $(V \cdot V^{-1})^{-1} = V \cdot V^{-1} \subseteq U \cdot U \subseteq W$. This completes the proof. □

The following theorem is now just a consequence of Theorem 1, Lemmas 3 and 4.

Theorem 2. *Let $(G, \cdot, \tau)$ be a semitopological group. If (G, τ) is a strongly Baire space then $(G, \cdot, \tau)$ is a topological group.*

4. Strongly Baire topological spaces

Although the class of strongly Baire spaces provided a convenient framework for our theorems in Sections 2 and 3 these spaces are, unfortunately, not readily identifiable. So in this section we will introduce a related class of spaces whose membership properties are more readily determined. Let (X, τ) be a topological space and let Y be a dense subset of X. On X we will consider the $\mathcal{G}(Y)$*-game* played between two players α and β. The rules for playing this game are the same as for the $\mathcal{G}_S(Y)$-game played on X with the only difference being in the definition of a win. In the $\mathcal{G}(Y)$-game we will say that α *wins* a play $((A_n, B_n)\colon n \in \mathbb{N})$ if, $\bigcap_{n\in\mathbb{N}} A_n \cap Y \neq \emptyset$. Otherwise the player β is said to have won this play. It follows in a similar manner to Theorems 1 and 2 in [17] that Y is *everywhere second category* in X (that is, $U \cap Y$ is a second category set in X for each non-empty open subset U of X, or equivalently, Y is a dense subset of X and a Baire space with the relative topology) if and only if the player β does not have a winning strategy in the $\mathcal{G}(Y)$-game. We will say that a topological space (Y, τ) is *cover semi-complete* is there exists a pseudo-metric d on Y such that;

(i) each d-convergent sequence in Y has a cluster-point in Y;
(ii) Y is *fragmented* by d, that is, for each $\varepsilon > 0$ and non-empty subset A of Y there exists a non-empty relatively open subset B of A such that d-diam $B < \varepsilon$ (see [15] for the original definition in terms of exhaustive covers).

Theorem 3. *If (X, τ) is a regular topological space that contains, as an everywhere second category set, a cover semi-complete space Y, then the player β does not have a winning strategy in the $\mathcal{G}_S(Y)$-game played on G. In particular, (X, τ) is a strongly Baire space.*

Proof. Let $t := (t_n\colon n \in \mathbb{N})$ be a strategy for the player β in the $\mathcal{G}_S(Y)$-game played on X. We need to construct a t-sequence $(A_n\colon n \in \mathbb{N})$ where α wins. To do this we will define a new strategy $t' := (t'_n\colon n \in \mathbb{N})$ for the player β in the $\mathcal{G}(Y)$-game played on X.

Step 1. Define $t'_1(\emptyset) := t_1(\emptyset)$. Now suppose that t'_j has been defined for each t'-sequence $(A_1, \dots, A_{j-1})$ of length $(j-1)$, $1 < j \leqslant n$ so that,

(i) $(A_1, \dots, A_{j-1})$ is a t-sequence;

(ii) $t'_j(A_1, \dots, A_{j-1}) \subseteq t_j(A_1, \dots, A_{j-1})$;

(iii) d-diam $(t'_j(A_1, \dots, A_{j-1}) \cap Y) < 1/j$.

Step $n+1$. For each t'-sequence $(A_1, \dots, A_n)$ of length n we define $t'_{n+1}(A_1, \dots, A_n)$ to be any non-empty open subset of $t_{n+1}(A_1, \dots, A_n)$ such that d-diam $(t'_{n+1}(A_1, \dots, A_n) \cap Y) < 1/n$. Note: this is possible since $(A_1, \dots, A_n)$ is a t-sequence and Y is "*fragmented*" by d. Hence with this definition,

(i) $(A_1, \dots, A_n)$ is a t-sequence;

(ii) $t'_{n+1}(A_1, \dots, A_n) \subseteq t_{n+1}(A_1, \dots, A_n)$;

(iii) d-diam $(t'_{n+1}(A_1, \dots, A_n) \cap Y) < 1/(n+1)$.

This completes the definition of $t' := (t'_n\colon\ n \in \mathbb{N})$. Now since Y is everywhere second category in X, t' cannot be a winning strategy for the player β in the $\mathcal{G}(Y)$-game played on X. Hence there is a t'-sequence (and so a t-sequence) $(A_n\colon\ n \in \mathbb{N})$ where $\bigcap_{n\in\mathbb{N}} A_n \cap Y \neq \emptyset$. It should now be clear that every sequence $(y_n\colon\ n \in \mathbb{N})$ in Y with $y_n \in A_n$ has a cluster-point in Y (and so in X). This shows that t is not a winning strategy for the player β in the $\mathcal{G}_S(Y)$-game played on X. In particular, X is a strongly Baire space. $\square$

We will say that a subset Y of a topological space (X, τ) has *countable separation in* X if there is a countable family $\{O_n\colon\ n \in \mathbb{N}\}$ of open subsets of X such that for every pair $\{x, y\}$ with $y \in Y$ and $x \in X \setminus Y$, $\{x, y\} \cap O_n$ is a singleton for at least one $n \in \mathbb{N}$. If we denote by, X_Σ the family of all subsets of X with countable separation in X then X_Σ is a σ-algebra that contains all the open subsets of X. Moreover, X_Σ is closed under the Souslin operation. For a completely regular topological space (X, τ) we shall say that X has *countable separation* if in some compactification bX, X has countable separation in bX. It is shown in [11] that if X has countable separation in one compactification then X has countable separation in every compactification and so we see that every Čech-analytic space has countable separation. Now Lemma 3.4 in [15] shows that each completely regular space with countable separation is cover semi-complete. In fact with a little extra work one can show that each $(p-\sigma)$-fragmentable space (see [2] for the definition) is cover semi-complete. Thus, it follows that the next corollary improves upon the main result of [3].

Corollary 1. *Let $(G, \cdot, \tau)$ be a regular semitopological group. If (G, τ) contains, as a second category subset, a cover semi-complete space Y, then $(G, \cdot, \tau)$ is a topological group. In particular, if (G, τ) is a cover semi-complete Baire space then $(G, \cdot, \tau)$ is a topological group.*

Proof. By Theorem 6.35 in [9] there exists a non-empty open subset U of G such that $U \cap Y$ is everywhere second category in U. Hence it is possible to construct a maximal disjoint family $\{U_\alpha\colon\ \alpha \in A\}$ of non-empty open subsets of G such that;

(i) $\bigcup_{\alpha\in A} U_\alpha$ is dense in G and

(ii) each U_α contains, as an everywhere second category subset, a cover semi-complete space D_α.

It is now easy to check that the set $D := \bigcup_{\alpha \in A} D_\alpha$ is everywhere second category in G and by appealing to the original definition of cover semi-completeness (given in [15]) it is routine to verify that the set D is itself cover semi-complete. Therefore, in light of Theorem 3, (G, τ) is a strongly Baire space and so the result follows from Theorem 2. □

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