A selection theorem for quasi-lower semicontinuous set-valued mappings

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Abstract:

Michael's Selection Theorems concern the existence of continuous selections for lower semicontinuous set-valued mappings. For quasi-lower semicontinuous set-valued mappings there are selection theorems which guarantee densely defined continuous selections. Here we present a theorem for quasi-lower semicontinuous set-valued mappings that produces a global selection which is continuous at the points of a residual subset of the domain. We also give applications to an extension of the Bartle-Graves Theorem and to minimal set-valued mappings.

Given a set-valued mapping Φ acting from a topological space X into nonempty subsets of a topological space Y, a *selection* for Φ is a single-valued mapping σ from X into Y with $\sigma(x) \in \Phi(x)$ for each $x \in X$. Selection theorems provide conditions under which there exists a continuous selection for a set-valued mapping. The best known of such theorems is Michael's Selection Theorem [10] which holds when X is a paracompact topological space and Φ is a lower semicontinuous mapping into non-empty closed convex subsets of a Banach space Y. Michael's Second Selection Theorem [11] restricts the domain X but reduces conditions on the range Y to be a complete metric space with Φ mapping into closed subsets of Y. Other Selection Theorems by Coban, Kenderov and Revalski [1] provide conditions under which there exists a dense G_δ subset X_1 of a Baire space X and a continuous selection σ for Φ defined on X_1. Their main result holds when Φ has a continuity property called "lower demicontinuity" and has a closed graph and Y is a complete metric space. A similar densely defined Selection Theorem [4] was given for a quasi-lower semicontinuous mapping Φ from a Baire space X into nonempty closed subsets of a Banach space Y. This result was later generalised in [9]. Here we present an improved Selection Theorem for a quasi-lower semicontinuous mapping Φ acting from a Baire space X into closed subsets of a topological space Y which has the Namioka property with respect to a complete metric ρ. We also consider the special case when Φ is a minimal set-valued mapping and establish a characterisation for the Namioka property in terms of residually continuous selections for minimal set-valued mappings.

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A set-valued mapping Φ acting from a topological space X into nonempty subsets of a topological space Y is said to be *lower semicontinuous* if for each $x_0 \in X$ and every open set W in Y with $\Phi(x_0) \cap W \neq \emptyset$ there exists an open neighbourhood U of x_0 such that $\Phi(x) \cap W \neq \emptyset$ for all $x \in U$. More generally, Φ is said to be *quasi-lower semicontinuous* if for each $x_0 \in X$ and every open set W in Y with $\Phi(x_0) \cap W \neq \emptyset$ there exists an open set U in X such that $x_0 \in \overline{U}$ and $\Phi(x) \cap W \neq \emptyset$ for all $x \in U$. We use the following obvious property of quasi-lower semicontinuous mappings.

Lemma 1. *Consider a quasi-lower semicontinuous mapping Φ acting from a topological space X into nonempty subsets of a topological space Y. For each pair of open sets U in X and V in Y with $\Phi(x) \cap V \neq \emptyset$ for all $x \in U$, the mapping $\Phi_{(U,V)}$ from U into subsets of V defined by,*

$$\Phi_{(U,V)}(x) := \Phi(x) \cap V$$

is a quasi-lower semicontinuous mapping from U into nonempty subsets of Y.

Consider a topological space (Y, τ) with ρ a metric on Y. We say that (Y, τ) has the *Namioka property with respect to ρ* if every τ-continuous mapping acting from a Baire space X into Y has at least one point of ρ-continuity. Equivalently, such a space has the property that every τ-continuous mapping acting from a Baire space X into Y is ρ-continuous at the points of a residual subset of X.

Our Namioka property extends that of Debs as given in [3, Lemma 7.2(ii)]. There they consider spaces $C(K)$, where K is compact, such that every continuous mapping acting from a Baire space X into $C(K)$, with the topology of pointwise convergence, is continuous with respect to the uniform norm at the points of a residual subset of X. Clearly such a space $C(K)$, with the topology of pointwise convergence, has the Namioka property according to our general definition. This special case of our definition has received alot of attention. In particular, the following results are known. If $C(K)$, with the topology of pointwise convergence, is σ-fragmented by the norm then $C(K)$ has the Namioka property, [5, Corol 3.13]. There exists a compact set K such that $C(K)$, with the topology of pointwise convergence, has the Namioka property but fails to be σ-fragmented by the norm, [13, Theorem 1.1]. There exist compact sets K such that $C(K)$, with the topology of pointwise convergence, fails to have the Namioka property, [14].

We use the following significant property of quasi-lower semicontinuous set-valued mappings.

Lemma 2. *Given a quasi-lower semicontinuous mapping Φ acting from a Baire space X into nonempty subsets of a topological space Y, the graph $G(\Phi) := \{(x, y) : x \in X, y \in \Phi(x)\}$*

is a Baire space with respect to the relative product topology.

Proof. Consider the mapping Ψ of X into subsets of $G(\Phi)$ defined by,

$$\Psi(x) := \{(x, y) : y \in \Phi(x)\}.$$

Suppose there exists a nonempty open subset V of $G(\Phi)$ such that $V \subseteq \bigcup\{C_n : n \in N\}$ where for each $n \in N$, C_n is a closed nowhere dense subset of $G(\Phi)$. Now Φ being quasi-lower semicontinuous implies that Ψ is quasi-lower semicontinuous so there exists a nonempty open subset U of X such that $\Psi(x) \cap V \neq \emptyset$ for all $x \in U$. By possibly replacing Ψ by $\Psi_{(U,V)}$ we may assume that $\Psi(U) \subseteq V$. Now for each $n \in N$, consider $O_n := \operatorname{int}\{x \in U : \Psi(x) \not\subseteq C_n\}$. Since Ψ is both quasi-lower semicontinuous and open each O_n is dense in U. Therefore, since X is a Baire space, $\bigcap\{O_n : n \in N\} \neq \emptyset$. However, for every $x \in \bigcap\{O_n : n \in N\}$ we have $\Psi(x) \not\subseteq \bigcup\{C_n : n \in N\}$, which contradicts the fact that $\Psi(U) \subseteq V \subseteq \bigcup\{C_n : n \in N\}$. Hence we may conclude that $G(\Phi)$ is a Baire space.

We use this property to show that topological spaces which have the Namioka property with respect to a metric have a fragmenting type property.

Lemma 3. *Consider a topological space (Y, τ) with a metric ρ on Y such that (Y, τ) has the Namioka property with respect to ρ and a τ-quasi-lower semicontinuous mapping Φ from a Baire space X into nonempty subsets of Y. Given $\varepsilon > 0$ and a nonempty open subset U of X there exist nonempty open subsets U' of U and V of Y such that $\Phi(x) \cap V \neq \emptyset$ for all $x \in U'$ and*

$$\rho - \operatorname{diam}[\Phi(U') \cap V] < \varepsilon.$$

Proof. By the previous Lemma $G(\Phi)$, with the relative product topology, is a Baire space. The projection mapping π from $G(\Phi)$ into Y defined by, $\pi((x, y)) := y$ is τ-continuous. Therefore, since (Y, τ) has the Namioka property, π is ρ-continuous at the points of a residual subset D of $G(\Phi)$.

Given $\varepsilon > 0$ and a nonempty open subset U of X, consider $(U \times Y) \cap G(\Phi)$, a nonempty open subset of $G(\Phi)$ and choose $(x, y) \in D \cap [(U \times Y) \cap G(\Phi)]$. Since π is ρ-continuous at (x, y) there exist open subsets U'' of U and V of Y such that $(x, y) \in U'' \times V$ and

$$\rho - \operatorname{diam}[\pi((U'' \times V) \cap G(\Phi))] < \varepsilon$$

and since Φ is τ-quasi-lower semicontinuous there exists a nonempty open subset U' of U'' such that $\Phi(x) \cap V \neq \emptyset$ for all $x \in U'$. Then we conclude that:

$$\rho - \operatorname{diam}[\Phi(U') \cap V] < \varepsilon.$$

Theorem 1. *Consider a topological space (Y, τ) with a complete metric ρ on Y such that (Y, τ) has the Namioka property with respect to ρ. Given a τ-quasi-lower semicontinuous set-valued mapping Φ acting from a Baire space X into nonempty subsets of Y there exists a function $\sigma : X \to Y$ with $\sigma(x) \in \overline{\Phi(x)}^{\rho}$ for each $x \in X$ that is ρ-continuous at the points of a residual subset of X.*

Proof. We begin by inductively constructing a sequence of families of orders pairs $F^n := \{(U^n_\alpha, \Phi^n_\alpha) : \alpha \in \Lambda_n\}$ consisting of nonempty open subsets $\{U^n_\alpha : \alpha \in \Lambda_n\}$ of X and τ-quasi-lower semicontinuous mappings $\{\Phi^n_\alpha : \alpha \in \Lambda_n\}$ such that for each $\alpha \in \Lambda_n$, Φ^n_α maps U^n_α into nonempty subsets of Y.

Our foundation step 0.
Let $\Lambda_0 := \{\emptyset\}, U^0_\emptyset := X$ and $\Phi^0_\emptyset := \Phi$ and define,

$$F^0 := \left\{(U^0_\alpha, \Phi^0_\alpha) : \alpha \in \Lambda_0\right\} \quad \text{and} \quad W^0 := \bigcup \{U^0_\alpha : \alpha \in \Lambda_0\} = X.$$

For each $n \in N$, we require the family F^n to have the following properties:

- (a_n) $U^n_\alpha \cap U^n_\beta = \emptyset$ for all $\alpha \neq \beta, \alpha, \beta \in \Lambda_n$.
- (b_n) $W^n := \bigcup \{U^n_\alpha : \alpha \in \Lambda_n\}$ is dense in X.
- (c_n) ρ–diam $\Phi^n_\alpha(U^n_\alpha) < \frac{1}{n}$ for all $\alpha \in \Lambda_n$.
- (d_n) For each $\alpha \in \Lambda_n$ there exists $\beta \in \Lambda_{n-1}$ such that $U^n_\alpha \subseteq U^{n-1}_\beta$ and $\Phi^n_\alpha(x) \subseteq \Phi^{n-1}_\beta(x)$ for all $x \in U^n_\alpha$.

Step 1. We construct F^1 satisfying these conditions. We partially order the families $\left\{(U^1_\alpha, \Phi^1_\alpha) : \alpha \in \Lambda_1\right\}$ satisfying properties (a_1) (c_1) and (d_1) with respect to set inclusion. Clearly, by Zorn's Lemma there exists a maximal family F^1 satisfying these conditions. We show that F^1 satisfies property (b_1). If $W^1 := \bigcup \{U^1_\alpha : \alpha \in \Lambda_1\}$ is not dense in X then there exists a nonempty open subset U of X such that $W^1 \cap U = \emptyset$. Now by Lemma 3 there exist nonempty open subsets U' of U and V of Y such that $\Phi(x) \cap V \neq \emptyset$ for all $x \in U'$ and ρ–diam $[\Phi(U') \cap V] < 1$. By Lemma 1, $\Phi_{(U',V)}$ is a τ-quasi-lower semicontinuous mapping of U' into subsets of Y. Clearly $(U', \Phi_{(U',V)}) \notin F^1$ and $\{(U', \Phi_{(U',V)})\} \cup F^1$ is a family satisfying the properties (a_1), (c_1) and (d_1) contradicting the maximality of Ω^1.

Assuming that we have produced the families F^k satisfying the properties (a_k), (b_k), (c_k) and (d_k) up to and including the nth step we proceed to construct the next step.

Step n+1. Consider $F^{n+1} := \{(U_\alpha^{n+1}, \Phi_\alpha^{n+1}) : \alpha \in \Lambda_{n+1}\}$ a family of ordered pairs satisfying the properties (a_{n+1}), (c_{n+1}) and (d_{n+1}) and maximal with respect to set inclusion. We show that F^{n+1} satisfies (b_{n+1}). If $W^{n+1} := \bigcup \{U_\alpha^{n+1} : \alpha \in \Lambda_{n+1}\}$ is not dense in X then there exists a nonempty open subset U of X such that $W^{n+1} \cap U = \emptyset$. Since W^n is dense in X, $W^n \cap U \neq \emptyset$ and so we may assume that $U \subseteq U_\beta^n$ for some $\beta \in \Lambda_n$. By Lemma 3 there exist nonempty open subsets U' of U and V of Y such that $\Phi_\beta^n(x) \cap V \neq \emptyset$ for all $x \in U'$ and ρ–diam $[\Phi_\beta^n(U') \cap V] < \frac{1}{n+1}$. By Lemma 1, $\Phi^n_{\beta(U',V)}$ is a τ-quasi-lower semicontinuous mapping of U' into subsets of Y. Clearly $(U', \Phi^n_{\beta(U',V)}) \notin F^{n+1}$ and $\{(U', \Phi^n_{\beta(U',V)})\} \cup F^{n+1}$ is a family satisfying the properties (a_{n+1}), (c_{n+1}) and (d_{n+1}) contradicting the maximality of Ω^{n+1}. This completes the inductive construction.

Now for each $n \in N$ we define the mapping Φ^n from W^n into subsets of Y by,

$$\Phi^n(x) := \Phi_\alpha^n(x) \quad \text{if} \quad x \in U_\alpha^n.$$

This mapping is well defined because our family $\{(U_\alpha^n, \Phi_\alpha^n) : \alpha \in \Lambda_n\}$ satisfies property (a_n). We now inductively construct a sequence of selections $\{\sigma_n\}$ for Φ that will be pointwise ρ–convergent to our desired mapping σ and which will be ρ–continuous at the points of $\bigcap\{W^n : n \in N\}$. Let σ_0 be any selection for Φ. Then for each $n \in N$ we define

$$\sigma_n(x) := \begin{cases} \sigma_n(x) \in \Phi^n(x) & \text{for } x \in W^n \\ \sigma_{n-1}(x) & \text{for } x \in X \backslash W^n \end{cases}$$

For $x \notin \bigcap\{W^n : n \in N\}$ the sequence $\{\sigma_n(x)\}$ will be eventually constant. By properties (c_n) and (d_n) and the fact that (Y, ρ) is complete, $\sigma(x) = \lim_{n\to\infty} \sigma_n(x)$ exists for each $x \in X$ and moreover, $\sigma(x) \in \overline{\Phi(x)}^\rho$ for each $x \in X$. Now for each $x \in \bigcap\{W^n : n \in N\}$ and $n \in N$ there exists an $\alpha \in \Lambda_n$ such that $x \in U_\alpha^n$. Therefore,

$$\sigma(U_\alpha^n) \subseteq \overline{\Phi_\alpha^n(U_\alpha^n)}^\rho \subseteq B\left[\sigma(x); \tfrac{1}{n}\right]$$

which shows that σ is continuous at x. So σ is the required single-valued mapping.

As an application of our Theorem we have an extension of Michael's result [11, Corol 1.4] which is related to the Bartle–Graves Theorem.

A mapping ψ acting from a topological space Y into a topological space X is said to be *quasi-open* if for each open set U in Y we have $\psi(U) \subseteq \overline{\text{int } \psi(U)}$. Clearly for such a mapping ψ the set-valued mapping $x \mapsto \psi^{-1}(x)$ from X into subsets of Y is quasi-lower semicontinuous on X. So we can make the following deduction.

Corollary. *Consider a topological space (Y, τ) with a complete metric ρ such that (Y, τ) has the Namioka property with respect to ρ. Given a quasi-open mapping ψ from (Y, τ)*

onto a Baire space X, if $\psi^{-1}(x)$ is ρ–closed for each $x \in X$, then there exists a mapping σ from X into Y which is ρ–continuous at the points of a residual subset of X such that

$$(\psi \circ \sigma)(x) = x \quad \textit{for all } x \in X.$$

A set-valued mapping Φ acting from a topological space X into nonempty subsets of a topological space Y is said to be *minimal* if for any open set W in Y and open set U in X such that $\Phi(U) \cap W \neq \emptyset$ there exists a nonempty open subset V of U such that $\Phi(V) \subseteq W$. Clearly such a minimal set-valued mapping Φ is quasi-lower semicontinuous on X.

However, for minimal set-valued mappings into topological spaces with the Namioka property we have a result stronger than Theorem 1 which is also an extension of Theorem 3.1 of [6] and of Theorem 3 of [8].

Theorem 2. *Consider a topological space (Y, τ) with a metric ρ such that (Y, τ) has the Namioka property with respect to ρ. A τ–minimal set-valued mapping Φ acting from a Baire space X into nonempty subsets of Y is single-valued and ρ–upper semicontinuous at the points of a residual subset of X.*

Proof. From Lemma 3 we have that given any $\varepsilon > 0$ and any nonempty open subset U of X there exist nonempty open subsets U' of U and V of Y such that $\Phi(x) \cap V \neq \emptyset$ for all $x \in U'$ and $\rho - \text{diam}\, [\Phi(U') \cap V] < \varepsilon$. Since Φ is τ–minimal there exists a nonempty open subset U'' of U' such that $\Phi(U'') \subseteq V$. Therefore, $\rho - \text{diam}\, \Phi(U'') < \varepsilon$. Writing $O_\varepsilon := \bigcup \{\text{open subsets } U \text{ of } X : \rho - \text{diam}\, \Phi(U) < \varepsilon\}$ we deduce that O_ε is open and dense in X and conclude that Φ is single-valued and ρ–upper semicontinuous at the points of $\bigcap \{O_{1/n} : n \in N\}$, a residual subset of X.

We note that a topological space (Y, τ) with a metric ρ such that (Y, τ) has the Namioka property with respect to ρ is actually characterised by all τ–minimal set-valued mappings acting from a Baire space into nonempty subsets of Y being single-valued and ρ–upper semicontinuous at the points of a residual subset of their domain.

In paper [12] we called a Banach space X a *general generic continuity space* if every weakly minimal mapping Φ from a complete metric space A into subsets of X is single-valued and norm upper semicontinuous at the points of a residual subset of A. A general generic continuity space X can be characterised by the property that every weakly quasi-continuous mapping from an α–favourable space A into X is norm continuous at the points of a residual subset of A, [7, Corol 3]. It follows then, from Theorem 2, that Banach space X where (X, weak) has the Namioka property with respect to the norm is a general generic continuity space. In the paper [5, Example 7.5] it was shown that (ℓ_∞, weak) is not σ–fragmentable while in [12, Corol 5.3] it was shown that ℓ_∞ is not a general generic continuity space. So we deduce that (ℓ_∞, weak) does not have the Namioka property with respect to the norm, a result first noted by Deville, [2].

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