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USCO SELECTIONS OF DENSELY DEFINED SET-VALUED MAPPINGS

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A set-valued mapping $\Phi : X \to 2^Y$ acting between topological spaces X and Y is said to be "lower demicontinuous" if the interior of the closure of the set $\Phi^{-1}(V) := \{x \in X : \Phi(x) \cap V \neq \emptyset\}$ is dense in the closure of $\Phi^{-1}(V)$ for each open set V in Y. Čoban, Kenderov and Revalski (1994) showed that for every densely defined lower demicontinuous mapping Φ acting from a Baire space X into subsets of a monotonely Čech-complete space Y, there exist a dense and G_δ subset $X_1 \subseteq X$ and an usco mapping $G : X_1 \to 2^Y$ such that $G(x) \subseteq \Phi^*(x)$, for every $x \in X_1$, where the mapping $\Phi^* : X \to 2^Y$ is the extension of Φ defined by,

$$\Phi^*(x) := \bigcap\{\overline{\Phi(W)} : W \text{ is a neighbourhood of } x\}.$$

In this paper we present a proof of the above result with the notion of monotone Čech-completeness replaced by the weaker notion of partition completeness. In addition, we observe that if the range space also lies in Stegall's class then we may assume that the mapping G is single-valued on X_1.

1. INTRODUCTION

Selection theorems provide conditions under which there exists a continuous selection for a set-valued mapping. In a recent paper [**4**], on selection theorems the authors presented a selection theorem for quasi-lower semicontinuous mappings that map from Baire spaces into subsets of topological spaces that are fragmented by complete metrics. In this paper we improve this result by presenting a selection theorem for "lower demicontinuous" mappings that map from Baire spaces into partition complete spaces. Specifically, we show that for a lower demicontinuous mapping Φ with closed graph acting from a Baire space X into a partition complete space Y there exist a dense and G_δ subset $X_1 \subseteq X$ and an usco mapping $G : X_1 \to 2^Y$ such that $G(x) \subseteq \Phi(x)$ for all $x \in X_1$. In addition we show that if the range space Y is partition complete and lies in Stegall's class then the mapping G may also be assumed to be single-valued on X_1. We also show that if the domain space X is α-favourable and the range space is partition complete and

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belongs to the class of weakly Stegall spaces then the mapping G is single-valued on an everywhere second category subset of X.

We end this section by giving some definitions, then in Section 2 we present the main result and finally in Section 3 we give some applications of our selection theorem.

Let (Y, τ) be a regular topological space, endowed with a pseudo-metric d. A filter-base $\mathcal{F}$ on Y is said to be *d-Cauchy* if for each $\varepsilon > 0$ there exists an $F \in \mathcal{F}$ such that $d - \operatorname{diam}(F) < \varepsilon$ and the space itself is said to be *partition complete* if the pseudo-metric d satisfies the following properties:

(i) every d-Cauchy filter-base $\mathcal{F}$ on Y has a τ-cluster point in Y (that is, $\bigcap\{\overline{F} : F \in \mathcal{F}\} \neq \emptyset$);

(ii) Y is "fragmented" by d, that is, for every $\varepsilon > 0$ and every non-empty subset A of Y there exists a τ-open subset B of Y such that $A \cap B \neq \emptyset$ and $d - \operatorname{diam}(A \cap B) < \varepsilon$.

(Note: It follows from (i) that in a partition complete space $\bigcap\{\overline{F} : F \in \mathcal{F}\}$ is non-empty and compact for every d-Cauchy filter-base $\mathcal{F}$.) The class of partition complete spaces is quite large including all the Čech complete spaces. More details on partition completeness can be found in [5].

2. Selection Theorem

Let $\Phi : X \to 2^Y$ be a set-valued mapping acting from a topological space X into subsets of a topological space Y. We call the mapping Φ *lower demicontinuous* on X if for every open set V in Y, the interior of the closure of the set $\Phi^{-1}(V) := \{x \in X : \Phi(x) \cap V \neq \emptyset\}$ is dense in the closure of $\Phi^{-1}(V)$, that is, $\operatorname{int}(\overline{\Phi^{-1}(V)})$ is dense in $\overline{\Phi^{-1}(V)}$. When $\{x \in X : \Phi(x) \neq \emptyset\}$ is dense in X, we say Φ is *densely defined.*

Lemma 1. *Consider a lower demicontinuous mapping Φ from a topological space X into subsets of a topological space Y. For each pair of non-empty open sets U in X and V in Y, the mapping $\Phi_{(U,V)}$ from U into subsets of V defined by, $\Phi_{(U,V)}(x) := \Phi(x) \cap V$ is a lower demicontinuous mapping on U.*

Proof: The proof of the lemma follows from the fact that for each open set $W \subseteq V$, $\Phi^{-1}_{(U,V)}(W) = \Phi^{-1}(W) \cap U$. ∎

A set-valued mapping $\Phi : X \to 2^Y$ acting between topological spaces X and Y is said to be an *usco* mapping if for each $x \in X$, $\Phi(x)$ is a non-empty compact subset of Y and for each open set W in Y, $\{x \in X : \Phi(x) \subseteq W\}$ is open in X.

Theorem 1. *Let X be a Baire space and Y be a Hausdorff partition complete space and let Φ be a densely defined lower demicontinuous set-valued mapping acting from X into subsets of Y. Then there exist a dense and G_δ-subset $X_1 \subseteq X$ and an usco mapping $G : X_1 \to 2^Y$ with $G(x) \subseteq \Phi^*(x)$ for all $x \in X_1$, where the mapping*

$\Phi^* : X \to 2^Y$ is defined by,

$$\Phi^*(x) := \bigcap\{\overline{\Phi(W)} : W \textit{ is a neighbourhood of } x\}.$$

In particular, $\{x \in X : \Phi^*(x) \neq \emptyset\}$ is residual in X.

PROOF: Let d be the fragmenting pseudo-metric on Y associated with the partition completeness of Y. To prove our theorem we inductively construct a sequence of families of ordered pairs $\mathcal{F}^n := \{(U_\alpha^n, \Phi_\alpha^n) : \alpha \in \Lambda^n\}$ consisting of non-empty open subsets $\{U_\alpha^n : \alpha \in \Lambda^n\}$ of X and densely defined lower demicontinuous mappings $\{\Phi_\alpha^n : \alpha \in \Lambda^n\}$ such that for each $\alpha \in \Lambda^n$, Φ_α^n maps U_α^n into subsets of Y.

BASE STEP. Consider $\Lambda^0 := \{\emptyset\}, U_\emptyset^0 := X$ and $\Phi_\emptyset^0 := \Phi$ and define,

$$\mathcal{F}^0 := \{(U_\alpha^0, \Phi_\alpha^0) : \alpha \in \Lambda^0\} \text{ and } W^0 := \bigcup\{U_\alpha^0 : \alpha \in \Lambda^0\} = X.$$

For each $n \in \mathbb{N}$, we require the family $\mathcal{F}^n$ to have the following properties:

- (a_n) $U_\alpha^n \cap U_\beta^n = \emptyset$ for each $\alpha \neq \beta$, $\alpha, \beta \in \Lambda^n$;
- (b_n) $W^n := \bigcup\{U_\alpha^n : \alpha \in \Lambda^n\}$ is dense in X;
- (c_n) $d - \operatorname{diam}[\Phi_\alpha^n(U_\alpha^n)] < 1/n$ for each $\alpha \in \Lambda^n$;
- (d_n) for each $\alpha \in \Lambda^n$ there exists a $\beta \in \Lambda^{n-1}$ such that $U_\alpha^n \subseteq U_\beta^{n-1}$ and $\Phi_\alpha^n(x) \subseteq \Phi_\beta^{n-1}(x)$ for each $x \in U_\alpha^n$.

STEP 1. Consider $\mathcal{F}^1 := \{(U_\alpha^1, \Phi_\alpha^1) : \alpha \in \Lambda^1\}$ a family of ordered pairs satisfying the properties $(a_1), (c_1)$ and (d_1) which is maximal with respect to set inclusion. By Zorn's lemma such a maximal family exists. We shall show that $\mathcal{F}^1$ satisfies property (b_1). If $W^1 := \bigcup\{U_\alpha^1 : \alpha \in \Lambda^1\}$ is not dense in X then there exists a non-empty open subset U of X such that $W^1 \cap U = \emptyset$. Since Y is fragmented by d and $\Phi_\emptyset^0$ is densely defined there exists an open set V in Y such that $\Phi_\emptyset^0(U) \cap V \neq \emptyset$ and $d - \operatorname{diam}[\Phi_\emptyset^0(U) \cap V] < 1$. By the lower demicontinuity of $\Phi_\emptyset^0$ on U there exists a non-empty open subset U' of U such that $(\Phi_\emptyset^0)_{(U',V)}$ is densely defined and lower demicontinuous on U' (by Lemma 1). Now $(U', (\Phi_\emptyset^0)_{(U',V)}) \notin \mathcal{F}^1$ and $\{(U', (\Phi_\emptyset^0)_{(U',V)})\} \cup \mathcal{F}^1$ is a family satisfying the properties $(a_1), (c_1)$ and (d_1). This contradicts the maximality of $\mathcal{F}^1$ and hence we may conclude that $\mathcal{F}^1$ satisfies property (b_1).

Assuming that we have constructed the families $\mathcal{F}^k$ in the sequence satisfying the properties $(a_k), (b_k), (c_k)$ and (d_k) up to and including the nth step, we proceed to construct the next step.

STEP $(n+1)$. Consider $\mathcal{F}^{n+1} := \{(U_\alpha^{n+1}, \Phi_\alpha^{n+1}) : \alpha \in \Lambda^{n+1}\}$ a family of ordered pairs satisfying the properties $(a_{n+1}), (c_{n+1})$ and (d_{n+1}) which is maximal with respect to set inclusion. We shall show that $\mathcal{F}^{n+1}$ satisfies property (b_{n+1}). If $W^{n+1} := \bigcup\{U_\alpha^{n+1} : \alpha \in \Lambda^{n+1}\}$ is not dense in X then there exists a non-empty open subset U of X such that $W^{n+1} \cap U = \emptyset$. Since W^n is dense in X, $W^n \cap U \neq \emptyset$ and so we may assume that $U \subseteq U_\beta^n$

for some $\beta \in \Lambda^n$. Now since Y is fragmented by d and Φ^n_β is densely defined there exists an open set V in Y such that $\Phi^n_\beta(U) \cap V \neq \emptyset$ and $d - \operatorname{diam}[\Phi^n_\beta(U) \cap V] < 1/(n+1)$. By the lower demicontinuity of Φ^n_β on U^n_β there exists a non-empty open subset U' of U such that $(\Phi^n_\beta)_{(U',V)}$ is densely defined and lower demicontinuous on U' (by Lemma 1). Clearly, $\left\{\left(U', (\Phi^n_\beta)_{(U',V)}\right)\right\} \notin \mathcal{F}^{n+1}$ and $\left\{\left(U', (\Phi^n_\beta)_{(U',V)}\right)\right\} \cup \mathcal{F}^{n+1}$ is a family satisfying the properties $(a_{n+1}), (c_{n+1})$ and (d_{n+1}). This contradicts the maximality of $\mathcal{F}^{n+1}$ and hence we may conclude that $\mathcal{F}^{n+1}$ satisfies property (b_{n+1}). This completes the inductive step.

Let $X_1 := \bigcap_{n=1}^{\infty} W^n$. Clearly X_1 is a dense-G_δ subset of X and for each $x \in X_1$ and $n \in \mathbb{N}$ there exists a unique $\alpha_n(x) \in \Lambda^n$ such that $x \in U^n_{\alpha_n(x)}$. Therefore we can define a set-valued mapping $\Psi : X_1 \to 2^Y$ by,

$$\Psi(x) := \bigcap_{n=1}^{\infty} \overline{\Phi^n_{\alpha_n(x)}(U^n_{\alpha_n(x)})}.$$

Clearly, Ψ is non-empty and compact-valued since for each $x \in X_1$,

$$\mathcal{F}(x) := \left\{\Phi^n_{\alpha_n(x)}\left(U^n_{\alpha_n(x)}\right) : n \in \mathbb{N}\right\}$$

is a d-Cauchy filter-base on Y. So to show that Ψ is an usco, it remains to show that Ψ is upper semicontinuous. To this end, consider $x \in X_1$ and O an open set containing $\Psi(x)$. Since $\Psi(x)$ is compact it will suffice to show that there exists an open neighbourhood U of x such that $\Psi(U) \subseteq \overline{O}$. We claim that for some $n_0 \in \mathbb{N}$, $\Phi^{n_0}_{\alpha_{n_0}(x)}\left(U^{n_0}_{\alpha_{n_0}(x)}\right) \subseteq O$, for otherwise, $\mathcal{F}^*(x) := \left\{\Phi^n_{\alpha_n(x)}\left(U^n_{\alpha_n(x)}\right) \setminus O : n \in \mathbb{N}\right\}$ would be a d-Cauchy filter-base on Y which would have a cluster point in $Y \setminus O$. But this is impossible since,

$$\emptyset \neq \bigcap_{F \in \mathcal{F}^*} \overline{F} \subseteq \bigcap_{F \in \mathcal{F}} \overline{F} = \Psi(x) \subseteq O.$$

Therefore there is some $n_0 \in \mathbb{N}$ such that $\Phi^{n_0}_{\alpha_{n_0}(x)}\left(U^{n_0}_{\alpha_{n_0}(x)}\right) \subseteq O$ and so,

$$\Psi(y) = \bigcap_{n=1}^{\infty} \overline{\Phi^n_{\alpha_n(y)}\left(U^n_{\alpha_n(y)}\right)} \subseteq \overline{\Phi^{n_0}_{\alpha_{n_0}(y)}\left(U^{n_0}_{\alpha_{n_0}(y)}\right)} = \overline{\Phi^{n_0}_{\alpha_{n_0}(x)}\left(U^{n_0}_{\alpha_{n_0}(x)}\right)} \subseteq \overline{O}$$

for all $y \in U^{n_0}_{\alpha_{n_0}(x)} \cap X_1$.

We now define the mapping $G : X_1 \to 2^Y$ by, $G(x) := \Psi(x) \cap \Phi^*(x)$ for all $x \in X_1$. We claim that the mapping G is an usco. Obviously G has a closed graph as both Ψ and Φ^* have closed graphs. Moreover, as $\operatorname{Gr}(G) \subseteq \operatorname{Gr}(\Psi)$ and Ψ is an usco, we have that G is also an usco (see, [**1**, page 309]), provided we can show that G has non-empty images. So in order to obtain a contradiction, let us suppose that for some $x_0 \in X_1$, $G(x_0) = \emptyset$. This means that the non-empty compact set $\{x_0\} \times \Psi(x_0)$ does not intersect the graph of

Φ^*. Since $\mathrm{Gr}(\Phi^*)$ is a closed subset of $X \times Y$, a straight forward compactness argument shows that there are open sets U of X and V of Y such that $x_0 \in U$, $\Psi(x_0) \subseteq V$ and $(U \times V) \cap \mathrm{Gr}(\Phi^*) = \emptyset$. Since $\Psi(x_0) \subseteq V$ it follows, as above, that there exists an $n_0 \in \mathbb{N}$ such that $\Phi^{n_0}_{\alpha_{n_0}(x_0)}(U^{n_0}_{\alpha_{n_0}(x_0)}) \subseteq V$ and so,

$$\emptyset \neq \Phi^{n_0}_{\alpha_{n_0}(x_0)}(U^{n_0}_{\alpha_{n_0}(x_0)} \cap U) \subseteq \Phi(U^{n_0}_{\alpha_{n_0}(x_0)} \cap U) \cap V \subseteq \Phi^*(U) \cap V = \emptyset.$$

This gives us the desired contradiction. Therefore G is an usco selection of Φ^*. ∎

REMARK 1. In Theorem 1, Φ^* is the unique mapping whose graph is the closure of the graph of Φ in $X \times Y$ (endowed with the product topology). In particular, if Φ has a closed graph then $\Phi^* = \Phi$.

An usco mapping Φ from a topological space X into subsets of a topological space Y is called a *minimal usco* if its graph does not strictly contain the graph of any other usco defined on X. We say that a topological space Y belongs to *Stegall's class* if for every Baire space X and minimal usco mapping $\Phi : X \to 2^Y$, Φ is single-valued at the points of a residual subset of X. A topological space Y belongs to the class of *weakly Stegall spaces* if for every α-favourable space X and minimal usco mapping $\Phi : X \to 2^Y$, Φ is single-valued at the points of an everywhere second category subset of X. For the sake of completeness we recall the definition of α-favourability.

Let X be a topological space. On X we consider the *Banach-Mazur game* played between two players α and β. A *play* of the game is a decreasing sequence of, alternately chosen, non-empty open subsets $A_n \subseteq B_n \subseteq \dots B_2 \subseteq A_1 \subseteq B_1$, where the sets A_n are chosen by player α and the sets B_n by player β. Player α is said to have *won* a play of the game if $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$. Otherwise player β is said to have won the play. A *strategy* s for player α is a rule that tells him or her how to play (possibly depending on all the previous moves of player β). Since the moves of player α may depend on the moves of player β, we denote the nth move of player α by, $s(B_1, B_2, \dots, B_n)$. We say that s is a *winning strategy*, if using it, player α wins every play, independently of the moves of player β. More information on Banach-Mazur game can be found in [**6**].

COROLLARY 1. *Let X be a Baire (an α-favourable) space and Y be a partition complete space that lies in Stegall's class (the class of weakly Stegall spaces). Suppose that $\Phi : X \to 2^Y$ is a densely defined lower demicontinuous mapping with closed graph. Then there exist a residual (everywhere second category) set $X_1 \subseteq X$ and a continuous selection $\sigma : X_1 \to Y$ of Φ on X_1.*

PROOF: First we shall consider the case when X is a Baire space, Y is partition complete and in Stegall's class. From Theorem 1 there exists an usco mapping $G : R \to 2^Y$ acting from a residual subset R of X into Y such that $G(x) \subseteq \Phi(x)$ for all $x \in R$. As every usco mapping contains a minimal usco mapping (see, [**2**, page 649]), the mapping G contains a minimal usco mapping $S : R \to 2^Y$. Now since the range space Y belongs to Stegall's class the mapping S is single-valued on a residual subset $X_1 \subseteq R$. The restriction of the mapping S to the set X_1 gives rise to the desired selection of Φ on X_1. In the case when the space Y belongs to the class of weakly Stegall spaces and X is α-favourable the proof follows in a similar fashion except that one requires the additional fact that a residual subset of an α-favourable space is again α-favourable. ∎

3. Applications

We say that a mapping $f : X \to Y$ from a topological space X into a topological space Y is *demi-open* if for every open set U in X the set $\operatorname{int}\overline{f(U)}$ is dense in $\overline{f(U)}$. It is easy to verify that $f^{-1} : Y \to 2^X$ is lower demicontinuous on Y if the mapping $f : X \to Y$ is demi-open on X.

Corollary 2. *Let $f : X \to Y$ be a demi-open mapping with closed graph acting from a partition complete space X which lies in Stegall's class (the class of weakly Stegall spaces) into a dense subset of a Baire space (an α-favourable space) Y. Then there exists a continuous mapping σ from a residual (everywhere second category) subset $Y_1 \subseteq Y$ into X such that $(f \circ \sigma)(x) = x$ for all x in Y_1.*

Proof: Let us consider the inverse mapping $f^{-1} : Y \to 2^X$. This is a densely defined lower demicontinuous mapping with closed graph. Hence from Corollary 1, there exist a residual (everywhere second category) subset $Y_1 \subseteq Y$ and a continuous selection $\sigma : Y_1 \to X$ of f^{-1} on Y_1. It follows then that $(f \circ \sigma)(x) = x$ for all $x \in Y_1$. ∎

Corollary 3. *Let $h : G \to K$ be a homomorphism acting from a partition complete group G into a Baire topological group K. If h is demi-open, has a closed graph and dense range then the mapping is open and onto K.*

Proof: The inverse mapping $h^{-1} : K \to 2^G$ is densely defined and lower demicontinuous with closed graph. Hence by Theorem 1 the domain of h^{-1} is residual in K, that is, the range of h is residual in K. However, as $h(G)$ is a subgroup of K it must be the case that $h(G) = K$. To show that h is open it suffices to show that for each non-empty open set U in G, $h(U)$ is somewhere residual in K and this follows by applying Theorem 1 to the inverse of the restriction of h to U. ∎

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