Separate and joint continuity of homomorphisms defined on topological groups

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Abstract. In this paper we prove the following theorem. “Let H be a strongly Baire topological group, X be a topological space and $(G, \cdot, \tau)$ be a topological group. If $f : H \times X \to G$ is a separately continuous mapping with the property that for each $x \in X$, the mapping $h \mapsto f(h, x)$ is a group homomorphism and D is a dense subset of X then for each q_D-point $x_0 \in X$ the mapping f is jointly continuous at each point of $H \times \{x_0\}$.” We also present some applications of this result.

AMS (2002) subject clasification: Primary 54C05, 22A10; Secondary 54E52, 39B99.

Keywords: Separate continuity; joint continuity; homomorphism; group; R-module.

1 Introduction

In this short note we prove a theorem significantly more general than the following. “Let H and G be topological groups and let X be a topological space. If H is Čech-complete (*i.e.*, a G_δ subset of its Stone-Čech compactification), X is a q-space and $f : H \times X \to G$ is a separately continuous mapping that possesses the property that for each $x \in X$, the mapping $h \mapsto f(h, x)$ is a group homomorphism, then f is jointly continuous on $H \times X$.”

We begin with some definitions. If X, Y and Z are topological spaces and $f : X \times Y \to Z$ is a function then we say that f is *quasi-continuous with respect to* Y *at* (x, y) if for each neighbourhood W of $f(x, y)$ and each product of open sets $U \times V \subseteq X \times Y$ containing (x, y) there exists a non-empty open subset $U' \subseteq U$ and a neighbourhood $V'$of y such that $f(U' \times V') \subseteq W$ and we say that f is *separately continuous on* $X \times Y$ if for each $x_0 \in X$ and $y_0 \in Y$ the functions $y \mapsto f(x_0, y)$ and $x \mapsto f(x, y_0)$ are both continuous on Y and X respectively.
Our contribution to this problem is based upon the following game. Let (X, τ) be a topological space and let D be a dense subset of X. On X we consider the $\mathcal{G}_S(D)$*-game* played between two players α and β. Player β goes first (always!) and chooses a non-empty open subset $B_1 \subseteq X$. Player α must then respond by choosing a non-empty open subset $A_1 \subseteq B_1$. Following this, player β must select another non-empty open subset $B_2 \subseteq A_1 \subseteq B_1$ and in turn player α must again respond by selecting a non-empty open subset $A_2 \subseteq B_2 \subseteq A_1 \subseteq B_1$. Continuing this procedure indefinitely the players α and β produce a sequence $((A_n, B_n) : n \in \mathbb{N})$ of pairs of open sets called a *play* of the $\mathcal{G}_S(D)$-game. We shall declare that α *wins* a play $((A_n, B_n) : n \in \mathbb{N})$ of the $\mathcal{G}_S(D)$-game if; $\bigcap_{n \in \mathbb{N}} A_n$ is non-empty and each sequence $(a_n : n \in \mathbb{N})$ with $a_n \in A_n \cap D$ has a cluster-point in X. Otherwise the player β is said to have won this play. By a *strategy* t for the player β we mean a *‘rule’* that specifies each move of the player β in every possible situation. More precisely, a strategy $t := (t_n : n \in \mathbb{N})$ for β is a sequence of τ-valued functions such that $t_{n+1}(A_1, \,.\,.\, A_n) \subseteq A_n$ for each $n \in \mathbb{N}$. The domain of each function t_n is precisely the set of all finite sequences $(A_1, A_2, \,.\,.\, A_{n-1})$ of length $n - 1$ in τ with $A_j \subseteq t_j(A_1, \,.\,.\, A_{j-1})$ for all $1 \leq j \leq n - 1$. (Note: the sequence of length 0 will be denoted by $\emptyset$.) Such a finite sequence $(A_1, A_2, .\,.\,.\, A_{n-1})$ or infinite sequence $(A_n : n \in \mathbb{N})$ is called a *t-sequence*. A strategy $t := (t_n : n \in \mathbb{N})$ for the player β is called a *winning strategy* if each infinite t-sequence is won by β. We will call a topological space (X, τ) a *strongly Baire* or

(*strongly β-unfavourable*) space if it is regular and there exists a dense subset D of X such that the player β does **not** have a winning strategy in the $\mathcal{G}_S(D)$-game played on X. It follows from Theorem 1 in [9] that each strongly Baire space is in fact a Baire space and it is easy to see that each strongly Baire space has at least one q_D-point. Indeed, if $t := (t_n : n \in \mathbb{N})$ is any strategy for β then there is a t-sequence $(A_n : n \in \mathbb{N})$ where α wins. In this case we have that each point of $\bigcap_{n\in\mathbb{N}} A_n$ is a q_D-point. Recall that a point $x \in X$ is called a q_D-point (with respect to some dense subset D of X) if there exists a sequence of neighbourhoods $(U_n : n \in \mathbb{N})$ of x such that every sequence $(x_n : n \in \mathbb{N})$ with $x_n \in U_n \cap D$ has a cluster-point in X. A q_X-point is usually just called a q-point. For more information on strongly Baire spaces see [5, Section 4].

2 Main Result

We shall use the following key results.

Lemma 1 [5, Lemma 1] *Let H be a strongly Baire space, X a topological space and G a regular space. If $f : H \times X \to G$ is a separately continuous mapping and D is a dense subset of X then for each q_D-point $x_0 \in X$ the mapping f is quasi-continuous with respect to X at each point of $H \times \{x_0\}$.*

Lemma 2 [6, Lemma 3.2] *Let $(G, \cdot, \tau)$ be a topological group. Then the topology on G is determined by the set of all continuous left-invariant pseudo-metrics d on G for which the mapping $(G, \tau) \times (G, d) \to (G, d)$ defined by, $(g, h) \mapsto g \cdot h$ is continuous.*

Recall that a pseudo-metric d defined on a topological group $(G, \cdot, \tau)$ is call *left-invariant* if for each g, h and k in G, $d(kg, kh) = d(g, h)$ and it is called *continuous* if the topology generated by d is coarser than τ.

One immediate consequence of this Lemma is the Banach-Steinhaus [8, Theorem 2].

Proposition 1 (Banach-Steinhaus Theorem) *If $f : H \to G$ is a Baire-1 (i.e., the pointwise limit of a sequence of continuous functions) group homomorphism acting from a Baire topological group H into a topological group G then f is continuous.*

Proof: Let d be any continuous left-invariant pseudo-metric on G. In light of Lemma 2 it is sufficient to show that $f : H \to (G, d)$ is continuous. By Osgood's Theorem [4, p. 86] (*i.e.,* Baire-1 functions defined on Baire spaces and mapping into pseudo-metric spaces are continuous on dense G_δ subsets of their domains) there exists a residual set H_0 in H on which f is d-continuous. Let h be any element of H and let $h_0 \in H_0$. Then for any net $\{h_\alpha : \alpha \in D\}$ in H converging to h we have that $f(h_\alpha) = f(hh_0^{-1})f(h_0h^{-1}h_\alpha)$, with $f(h_0h^{-1}h_\alpha) \to f(h_0)$ in (G, d) since $h_0h^{-1}h_\alpha \to h_0$ in H. Therefore, $f(h_\alpha) \to f(hh_0^{-1})f(h_0) = f(h)$ in (G, d); which completes the proof. □

Theorem 1 *Let H be a strongly Baire topological group, X be a topological space and $(G, \cdot, \tau)$ be a topological group. If $f : H \times X \to G$ is a separately continuous mapping with the property that for each $x \in X$, the mapping $h \mapsto f(h, x)$ is a group homomorphism and D is a dense subset of X then for each q_D-point $x_0 \in X$ the mapping f is jointly continuous at each point of $H \times \{x_0\}$.*

Proof: Suppose $x_0 \in X$ is a q_D-point. As with the Banach-Steinhaus Theorem we will appeal to Lemma 2 to deduce that it will be sufficient to prove that for any continuous left-invariant pseudo-metric d on G for which that mapping $(G, \tau) \times (G, d) \to (G, d)$, defined by, $(g, h) \mapsto g \cdot h$ is continuous, the mapping $f : H \times X \to (G, d)$ is continuous at each point of $H \times \{x_0\}$. Fix $\varepsilon > 0$ and consider the open set:

$$O_\varepsilon := \bigcup\{\text{open sets } U \subseteq H : \text{ there is a neighbourhood } V \text{ of } x_0 \text{ with } d - \text{diam}[f(U \times V)] < \varepsilon\}.$$

We shall show that O_ε is dense in H. To this end, let U_0 be a non-empty open subset of H and let $h_0 \in U_0$. Since, by Lemma 1, f is quasi-continuous with respect to X there exists a non-empty open subset U of U_0 and a neightbourhood V of x_0 such that $f(U \times V) \subseteq B_d(f(h_0, x_0); \varepsilon/3)$. Therefore, $d - \text{diam}[f(U \times V)] < \varepsilon$ and so $\emptyset \neq U \subseteq O_\varepsilon \cap U_0$; which shows that O_ε is dense in H. Hence, f is d-continuous at each point of $(\bigcap_{n\in\mathbb{N}} O_{1/n}) \times \{x_0\}$; which is non-empty. We now show that f is in fact d-continuous at each point of $H \times \{x_0\}$. To this end, let h_0 be any element of $\bigcap_{n\in\mathbb{N}} O_{1/n}$ and let h be any element of H and suppose that $\{(h_\alpha, x_\alpha) : \alpha \in D\}$ is a net in $H \times X$ converging to (h, x_0). Then by using the fact that,

$$f(h_\alpha, x_\alpha) = f(hh_0^{-1}, x_\alpha) f(h_0 h^{-1} h_\alpha, x_\alpha)$$

and $h_0 h^{-1} h_\alpha \to h_0$ we obtain that $f(h_\alpha, x_\alpha) \to f(hh_0^{-1}, x_0) f(h_0, x_0) = f(h, x_0)$. [Note: we also used the fact that $f(hh_0^{-1}, x_\alpha) \to f(hh_0^{-1}, x_0)$ in (G, τ).] This proves that f is d-continuous at each point of $H \times \{x_0\}$; which in turn implies that f is continuous at each point of $H \times \{x_0\}$. $\square$

Remark: If H is Čech-complete then we may relax the hypothesis that "for each $x \in X$, $h \mapsto f(h, x)$ is continuous" to "for each $x \in X$, $h \mapsto f(h, x)$ is Souslin measurable" [7].

3 Applications

In this section we present a few sample applications of Theorem 1.

Let X and Y be arbitrary sets and let $A \subseteq X$ and $B \subseteq Y$. We shall write $F(A; B)$ for the set of all functions from X into Y that map A into B, that is,

$$F(A; B) := \{f \in Y^X : f(A) \subseteq B\}.$$

If X and Y are topological spaces then the *compact-open* (*pointwise*) topology on Y^X is the topology generated by the sets

$$\{F(A; B) : A \in \mathscr{A} \text{ and } B \in \mathscr{G}\}$$

where $\mathscr{A}$ denotes the class of compact subsets (singleton subsets) of X and $\mathscr{G}$ denotes the class of open subsets of Y.

For a topological group G we shall denote by $\text{End}_p(G)$ the space of all continuous endomorphisms on G endowed with the topology of pointwise converegence on G.

Corollary 1 *Suppose that G is a strongly Baire topological group. If Σ is a q_D-subspace of $End_p(G)$ for some dense subset D of Σ (i.e., each point in Σ is a q_D-point) then the mapping $\pi : End_p(G) \times \Sigma \to End_p(G)$, defined by $\pi(m, m') := m' \circ m$ is continuous. In particular, π is continuous on Σ.*

Proof: Consider the mapping $f : G \times \Sigma \to G$, defined by $f(g, m) := m(g)$. Then f satisfies the hypotheses of Theorem 1 and so is jointly continuous on $G \times \Sigma$. Now, if $\{(m_\alpha, m'_\alpha) : \alpha \in D\}$ is a net in $\text{End}_p(G) \times \Sigma$ converging to (m, m') and $g \in G$ then,

$$\pi(m_\alpha, m'_\alpha)(g) = m'_\alpha(m_\alpha(g)) \to m'(m(g)) = \pi(m, m')(g).$$

Hence π is continuous on $\text{End}_p(G) \times \Sigma$. $\square$

Let G and H be topological groups. We shall denote by $\text{Hom}_p(H; G)$ the space of all continuous homomorphisms from H into G endowed with the topology of pointwise convergence on H.

Corollary 2 *Let G and H be topological groups and let Σ be a subset of $Hom_p(H;G)$. If H is a strongly Bare space and Σ is a q_D-subspace of $Hom_p(H;G)$ for some dense subset D of Σ then on Σ the pointwise and compact-open topologies coincide.*

Proof: Since the compact-open topology is always finer than the pointwise topology it will be sufficient to show that for each compact set $K \subseteq H$ and open set $W \subseteq G$, $F(K;W)$ is open in the pointwise topology. To this end, let K be a non-empty compact subset of H, W be a non-empty open subset of G and $m_0 \in F(K;W)$ (*i.e.,* $m_0(K) \subseteq W$). From Theorem 1 it follows that the mapping $f : H \times \Sigma \to G$ defined by, $f(h,m) := m(h)$ is jointly continuous on $H \times \Sigma$ and so from a simple compactness argument it follows that there exists an open set U in H and an open neighbourhood V of m_0 in Σ such that $K \subseteq U$ and $f(U \times V) \subseteq W$. In particular this means that $m_0 \in V \subseteq F(K;W)$ and so $F(K;W)$ is open in the pointwise topology. □

If $(M,+)$ is an Abelian group endowed with a topology τ_1 and $(R,+,\cdot)$ is a ring endowed with a topology τ_2 then we say M is a *semitopological R-module* (over R) if it is an R-module (over R) and both the mappings $(x,y) \mapsto x+y$ and $(r,x) \mapsto r \cdot x$ are separately continuous on $M \times M$ and $R \times M$ respectively.

Corollary 3 *Let M be a semitopological R-module over a ring R. If R is a q_D-space for some dense subset D of R and M is a strongly Baire space then M is a topological R-module, i.e., $(M,+)$ is a topological group and $(r,x) \mapsto r \cdot x$ is jointly continuous.*

Proof: By Theorem 2 in [5], which states that every group endowed with a strongly Baire topology for which mulitplication is separately continuous is in fact a topological group, we may deduce that $(M,+)$ is a topological group. The result then follows for Theorem 1 since for each fixed $r \in R$ the mapping $x \mapsto r \cdot x$ is an endomorphism of M. □

Remark: If strongly Baire is replaced by Čech-complete (in M) and q_D-space is replaced by Čech-completeness (in R) then we may relax the hypothesis that $(r,x) \mapsto r \cdot x$ is separately continuous to $(r,x) \mapsto r \cdot x$ being separately Souslin measurable [7].

We say that a mapping $f : X \to Y$ acting between Banach spaces X and Y is *almost $\mathcal{C}^1$* on X if for each $y \in X$ the mapping $x \mapsto f'(x;y)$ defined by, $f'(x;y) := \text{weak-}\lim_{t \to 0}[f(x+ty) - f(x)]/t$ is norm-to-norm continuous on X.

Corollary 4 *Let $f : X \to Y$ be a continuous mapping acting between Banach spaces X and Y. If f is almost $\mathcal{C}^1$ on X then the mapping $(x,y) \mapsto f'(x;y)$ is jointly norm continuous on $X \times X$.*

Proof: Fix $x_0 \in X$. We shall show first that the mapping $y \mapsto f'(x_0;y)$ is linear. To do this it is sufficient to show that for each $y^* \in Y^*$ the mapping $y \mapsto y^*(f'(x_0;y))$ is linear on X. Fix $y^* \in Y^*$ and let $g : X \to \mathbb{R}$ be defined by, $g(x) := (y^* \circ f)(x)$, then g is continuous and almost $\mathcal{C}^1$ with $g'(x;y) = y^*(f'(x;y))$ for each $y \in X$. It now follows, as in the finite dimensional case, (see [1], p.261) that the mapping $y \mapsto g'(x_0;y)$ is linear on X [Note: $g'(x_0,y)$ is linear on X if it is linear on every 2 dimensional subspace of X]. Next, let $\{t_n : n \in \mathbb{N}\}$ be any sequence of positive numbers converging to 0 and define $f_n : X \to Y$ by, $f_n(y) := [f(x_0 + t_n y) - f(x_0)]/t_n$. Then each f_n is continuous and $\text{weak-}\lim_{n \to \infty} f_n(y) = f'(x_0;y)$ for each $y \in X$. Therefore by the Banach-Steinhaus Theorem $y \mapsto f'(x_0;y)$ is norm-to-weak continuous. Since $y \mapsto f'(x_0;y)$ is linear it follows from the uniformly boundedness theorem that $y \mapsto f'(x_0;y)$ is norm-to-norm continuous. The result then follows from Theorem 1. □

Remark: We say that a mapping $f : X \to Y$ acting between Banach spaces X and Y is *weakly $\mathcal{C}^1$* if for each $y \in X$ and $y^* \in Y^*$ the mapping $x \mapsto (y^* \circ f)'(x;y)$ is continuous on X. Now if Y is reflexive and for each fixed $x_0 \in X$ and $y \in X$ the mapping $T_y^{x_0} : Y^* \to \mathbb{R}$ defined by $T_y^{x_0}(y^*) := (y^* \circ f)'(x_0; y)$ is a bounded linear functional on Y^* (note: this mapping is necessarily linear) then weak-$\lim_{t\to 0}[f(x_0 + ty) - f(x_0)]/t$ exists. Moreover, by examining the proof of Corollary 4 we see that if f is norm-to-weak continuous and weakly $\mathcal{C}^1$ then for each $x_0 \in X$ such that each $T_y^{x_0}$ is bounded for all $y \in X$, the mapping $y \mapsto f'(x; y)$ is a bounded linear operator between X and Y.

References

[1] Tom M. Apostol, Calculus Vol. II: Multi-variable calculus and linear algebra, with applications to differential equations and probability. Second edition *Blaisdell Publishing Co. Ginn and Co. Waltham, Mass.-Toronto, Ont.-London* 1969.

[2] A. Bouziad, Continuity of separately continuous group actions in p-spaces, *Topology Appl.* **71** (1996), 119–124.

[3] G. Hansel and J. P. Troallic, Quasicontinuity and Namioka's theorem, *Topology Appl.* **46** (1992), 135–149.

[4] J. I. Kelley and I. Namioka, *Topological Linear Spaces,* Graduate Texts in Mathematics, No. 36. *Springer-Verlag, New York-Heidelberg,* 1976.

[5] P. S. Kenderov, I. S. Kortezov and W. B. Moors, Topological games and topological groups, *Topology Appl.* **109** (2001), 157–165.

[6] I. Namioka, Separate and joint continuity, *Pacific J. Math.* **51** (1974), 515–531.

[7] D. Noll, Souslin measurable homomorphisms of topological groups, *Arch. Math.* **59** (1992), 294–301.

[8] B. J. Pettis, On continuity and openness of homomorphisms in topological groups, *Ann. of Math.* **52** (1950), 293-308.

[9] Jean Saint Raymond, Jeux topologiques et espaces de Namioka, *Proc. Amer. Math. Soc.* **87** (1983), 499-504.