NON-SMOOTH ANALYSIS, OPTIMISATION THEORY AND BANACH SPACE THEORY

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Abstract. The questions listed here do not necessarily represent the most significant problems from the areas of Non-smooth Analysis, Optimisation theory and Banach space theory, but rather, they represent a selection of problems that are of particular interest to the authors.

1. Weak Asplund spaces

Let X be a Banach space. We say that a function $\varphi : X \to \mathbb{R}$ is *Gâteaux differentiable at* $x \in X$ if there exists a continuous linear functional $x^* \in X^*$ such that

$$x^*(y) = \lim_{\lambda \to 0} \frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda} \quad \text{for all } y \in X.$$

In this case, the linear functional x^* is called the *Gâteaux derivative of* φ *at* $x \in X$. If the limit above is approached uniformly with respect to all $y \in B_X$-the closed unit ball in X, then φ is said to be *Fréchet differentiable at* $x \in X$ and x^* is called the *Fréchet derivative of* φ *at* x.

A Banach space X is called a *weak Asplund space* [*Gâteaux differentiability space*] if each continuous convex function defined on it is Gâteaux differentiable at the points of a residual subset (i.e., a subset that contains the intersection of countably many dense open subsets of X) [dense subset] of its domain.

Since 1933, when S. Mazur [29] showed that every separable Banach space is weak Asplund, there has been continued interest in the study of weak Asplund spaces. For an introduction to this area see, [36] and [17]. Also see the seminal paper [1] by E. Asplund.

The main problem in this area is given next.

Question 1.1. *Provide a geometrical characterisation for the class of weak Asplund spaces.* **1001 ?**

Note that there is a geometrical dual characterisation for the class of Gâteaux differentiability spaces, see [39]. However, it has recently been shown that there are Gâteaux differentiability spaces that are not weak Asplund [33]. Hence the dual characterisation for Gâteaux differentiability spaces cannot serve as a dual characterisation for the class of weak Asplund spaces.

The description of the next two related problems requires some additional definitions.

Let $A \subseteq (0, 1)$ and let $K_A := [(0, 1] \times \{0\}] \cup [(\{0\} \cup A) \times \{1\}]$. If we equip this set with the order topology generated by the lexicographical (dictionary) ordering (i.e., $(s_1, s_2) \leq (t_1, t_2)$ if, and only if, either $s_1 < t_1$ or $s_1 = t_1$ and $s_2 \leq t_2$) then with this topology K_A is a compact Hausdorff space [26]. In the special case of $A = (0, 1)$, K_A reduces to the well-known "double arrow" space.

? 1002 **Question 1.2.** *Is $(C(K_A), \|\cdot\|_\infty)$ weak Asplund whenever A is perfectly meagre?*

Recall that a subset $A \subseteq \mathbb{R}$ is called *perfectly meagre* if for every perfect subset P of $\mathbb{R}$ the intersection $A \cap P$ is meagre (i.e., first category) in P. An affirmative answer to this question will provide an example (in ZFC) of a weak Asplund space whose dual space is not weak* fragmentable, see [33] for more information on this problem. For example, it is shown in [33] that if A is perfectly meagre then $(C(K_A), \|\cdot\|_\infty)$ is *almost weak Asplund* i.e., every continuous convex function defined on $(C(K_A), \|\cdot\|_\infty)$ is Gâteaux differentiable at the points of an everywhere second category subset of $(C(K_A), \|\cdot\|_\infty)$. Moreover, it is also shown in [33] that if $(C(K_A), \|\cdot\|_\infty)$ is weak Aplund then A is obliged to be perfectly meagre.

Our last question on this topic is the following well-known problem.

? 1003 **Question 1.3.** *Is $(C(K_{(0,1)}), \|\cdot\|_\infty)$ a Gâteaux differentiability space?*

The significance of this problem emanates from the fact that $(C(K_{(0,1)}), \|\cdot\|_\infty)$ is not a weak Asplund space as the norm $\|\cdot\|_\infty$ is only Gâteaux differentiable the the points of a first category subset of $(C(K_{(0,1)}), \|\cdot\|_\infty)$, [17]. Hence a positive solution to this problem will provide another example of a Gaâteaux differentiability space that is not weak Asplund.

2. Bishop-Phelps Problem

For a Banach space $(X, \|\cdot\|)$, with closed unit ball B_X, the *Bishop-Phelps set* is the set of all linear functionals in the dual X^* that attain their maximum value over B_X; that is, the set $\{x^* \in X^* : x^*(x) = \|x\| \text{ for some } x \in B_X\}$. The Bishop-Phelps theorem [3] says that the Bishop-Phelps set is always dense in X^*.

? 1004 **Question 2.1.** *Suppose that $(X, \|\cdot\|)$ is a Banach space. If the Bishop-Phelps set is a residual subset of X^* (i.e., contains, as a subset, the intersection of countably many dense open subsets of X^*) is the dual norm necessarily Fréchet differentiable on a dense subset of X^*?*

The answer to this problem is known to be positive in the following cases:

(i) if X^* is weak Asplund, [18, Corollary 1.6(i)];
(ii) if X admits an equivalent weakly mid-point locally uniformly rotund norm and the weak topology on X is σ-fragmented by the norm, [34, Theorem 3.3 and Theorem 4.4];
(iii) if the weak topology on X is Lindelöf, [28].

Condition (ii) can be slightly improved, see [19, Theorem 2]. It is also known that each equivalent dual norm on X^* is Fréchet differentiable on a dense subset on X^* whenever the Bishop-Phelps set of each equivalent norm on X is residual in X^*, [32, Theorem 4.4]. Note that in this case X has the Radon-Nikodým property.

For an historical introduction to this problem and its relationship to LUR renorming theory see, [27].

Next, we give an important special case of the previous question.

Question 2.2. *If the Bishop-Phelps set of an equivalent norm $\|\cdot\|$ defined on $(\ell^\infty(\mathbb{N}), \|\cdot\|_\infty)$ is residual, is the corresponding closed unit ball dentable?* **1005 ?**

Recall that a nonempty bounded subset A of a normed linear space X is *dentable* if for every $\varepsilon > 0$ there exists a $x^* \in X^* \setminus \{0\}$ and a $\delta > 0$ such that

$$\|\cdot\| - \operatorname{diam}\{a \in A : x^*(a) > \sup_{x \in A} x^*(x) - \delta\} < \varepsilon.$$

It is well-known that if the dual norm has a point of Fréchet differentiability then B_X is dentable [46].

3. Metrizability of compact convex sets

One facet of the study of compact convex subsets of locally convex spaces is the determination of their metrizability in terms of topological properties of their extreme points. For example, a compact convex subset K of a Hausdorff locally convex space X is metrizable if, and only if, the extreme points of K (denoted $\operatorname{Ext}(K)$) are Polish (i.e., homeomorphic to a complete separable metric space), [12].

Since 1970 there have been many papers on this topic (e.g. [12, 14, 25, 30, 41] to name but a few).

Question 3.1. *Let K be a nonempty compact convex subset of a Hausdorff locally convex space (over $\mathbb{R}$). Is K metrizable if, and only if, $A(K)$ - the continuous real-valued affine mappings defined on K, is separable with respect to the topology of pointwise convergence on* $\operatorname{Ext}(K)$*?* **1006 ?**

The answer to this problem is known to be positive in the following cases:

(i) if $\overline{\operatorname{Ext}(K)}$ is Lindelöf, [35];
(ii) if $\overline{\operatorname{Ext}(K)} \setminus \operatorname{Ext}(K)$ is countable, [35].

Question 3.1 may be thought of as a generalization of the fact that a compact Hausdorff space K is metrizable if, and only if, $C_p(K)$ is separable. Here $C_p(K)$ denotes the space of continuous real-valued functions defined on K endowed with the topology of pointwise convergence on K.

4. The Boundary Problem

Let $(X, \|\cdot\|)$ be a Banach space. A subset B of the dual unit ball $B_{X^*} := \{x^* \in X^*: \|x^*\| \leq 1\}$ is called a boundary if for any $x \in X$, there is $x^* \in B$ such that $x^*(x) = \|x\|$. A simple example of boundary is provided by the set $\mathrm{Ext}(B_{X^*})$ of extreme points of B_{X^*}. This notion came into light after James' characterization of weak compactness [24], and has been studied in several papers (e.g. [45, 42, 47, 20, 21, 10, 8, 7, 22, 9]. In spite of significant efforts, the following question is still open (see [20, Question V.2] and [16, Problem I.2]):

? 1007 **Question 4.1.** *A be a norm bounded and $\tau_p(B)$ compact subset of X. Is A weakly compact?*

The answer to the boundary problem is known to be positive in the following cases:

(i) if A is convex, [45];
(ii) if $B = \mathrm{Ext}(B_{X^*})$, [5];
(iii) if X does not contain an isomorphic copy of $l_1(\Gamma)$ with $|\Gamma| = \mathfrak{c}$, [8, 9];
(iv) if $X = C(K)$ equipped with their natural norm $\|\cdot\|_\infty$, where K is an arbitrary compact space, [7].

Case (i) can be also obtained from James' characterization of weak compactness, see [21]. The original proof for (ii) given in [5] uses, amongst other things, deep results established in [4]. Case (iii) is reduced to case (i): if $l_1(\Gamma) \not\subset X$, $|\Gamma| = \mathfrak{c}$, and $C \subset B_{X^*}$ is 1-norming (i.e., $\|x\| = \sup\{|x^*(x)| : x^* \in C\}$), it is proved in [8, 9] that for any norm bounded and $\tau_p(C)$-compact subset A of X, the closed convex hull $\overline{\mathrm{co}(A)}^{\tau_p(C)}$ is again $\tau_p(C)$-compact; the class of Banach spaces fulfilling the requirements in (iii) is a wide class of Banach spaces that includes: weakly compactly generated Banach spaces or more generally weakly Lindelöf Banach spaces and spaces with dual unit ball without a copy of $\beta\mathbb{N}$. The techniques used in case (iv) are somewhat different, and naturally extend the classical ideas of Grothendieck, [23], that led to the fact that norm bounded $\tau_p(K)$-compact subsets of spaces $C(K)$ are weakly compact. It should be noted that it is easy to prove that for any set Γ, the boundary problem has also positive answer for the space $\ell^1(\Gamma)$ endowed with its canonical norm, see [7, 9].

We observe that the solution in full generality to the boundary problem without the concourse of James' theorem of weak compactness would imply an alternative proof of the following version of James' theorem itself: a Banach space X is reflexive if, and only if, each element $x^* \in X^*$ attains its maximum in B_X.

Finally, we point out that in the papers [43, 50], it has been claimed that the boundary problem was solved in full generality. Unfortunately, to the best of our knowledge both proofs seem not to be correct.

5. Separate and Joint Continuity

If X, Y and Z are topological spaces and $f : X \times Y \to Z$ is a function then we say that f is *jointly continuous at* $(x_0, y_0) \in X \times Y$ if for each neighbourhood W

of $f(x_0, y_0)$ there exists a product of open sets $U \times V \subseteq X \times Y$ containing (x_0, y_0) such that $f(U \times V) \subseteq W$ and we say that f is *separately continuous on* $X \times Y$ if for each $x_0 \in X$ and $y_0 \in Y$ the functions $y \mapsto f(x_0, y)$ and $x \mapsto f(x, y_0)$ are both continuous on Y and X respectively.

Since the paper [2] of Baire first appeared there has been continued interest in the question of when a separately continuous function defined on a product of "nice" spaces admit a point (or many points) of joint continuity and over the years there have been many contributions to this area (e.g. [6, 11, 13, 15, 28, 38, 31, 40, 44, 48] etc.) Most of these results can be classified into one of two types. (I) The existence problem, i.e., if $f : X \times Y \to \mathbb{R}$ is separately continuous find conditions on either X or Y (or both) such that f has at least one point of joint continuity. (II) The fibre problem, i.e., if $f : X \times Y \to \mathbb{R}$ is separately continuous find conditions on either X or Y (or both) such that there exists a nonempty subset R of X such that f is jointly continuous at the points of $R \times Y$.

The main existence problem is, [49]:

Question 5.1. *Let X be a Baire space and let Y be a compact Hausdorff space. If $f : X \times Y \to \mathbb{R}$ is separately continuous does f have at least one point of joint continuity?* **1008 ?**

We will say that a Baire space X has the *Namioka Property* of has property $\mathcal{N}$ if for every compact Hausdorff space Y and every separately continuous function $f : X \times Y \to \mathbb{R}$ there exists a dense G_δ-subset G of X such that f is jointly continuous at each point of $G \times Y$. Similarly, we will say that a compact Hausdorff space Y has the *co-Namioka Property* or has property $\mathcal{N}^*$ if for every Baire space X and every separately continuous function $f : X \times Y \to \mathbb{R}$ there exists a dense G_δ-subset G of X such that f is jointly continuous at each point of $G \times Y$.

The main fibre problems are:

Question 5.2. *Characterize the class of Namioka spaces.* **1009 ?**

Question 5.3. *Characterize the class of co-Namioka spaces.* **1010 ?**

For a good introduction to this topic see, [31, 40].

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