Norm continuity of weakly continuous mappings into Banach spaces.*

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Abstract

Sigma-fragmentability of a Banach space E is equivalent to the existence of a winning strategy for one of the players in a topological "fragmenting" game in (E, weak). We show that the absence of a winning strategy for the other player is equivalent to each of the following properties:

(i) *for every continuous mapping $f : Z \to (E, weak)$, where Z is an α-favorable space, there exists a dense G_δ-subset $A \subset Z$ at the points of which f is norm continuous;*

(ii) *for every quasi-continuous mapping $f : Z \to (E, weak)$, where Z is a complete metric space, there exists a point at which f is weakly continuous;*

(iii) *for every quasi-continuous mapping $f : Z \to (E, weak)$, where Z is an α-favorable space, there exists a dense G_δ-subset $A \subset Z$ at the points of which f is norm continuous.*

Thus we provide an internal characterization of those Banach spaces that satisfy property (i). Moreover we show that similar properties hold for spaces of the type $C(T)$, endowed with the topology of pointwise convergence on T. From this we derive some results concerning joint continuity of functions $f(z,t)$ which are "quasi-separately continuous" on $Z \times T$.

For $E = l^\infty$ and $E = l^\infty/c_0$ we explicitly describe how to construct a weakly continuous nowhere norm continuous mapping $f : Z \to E$, where Z is some completely regular α-favorable space.

1 Introduction.

Let X be a topological space, ρ some metric on it (not necessarily generating the topology of X) and $\varepsilon > 0$. A subset $X_1 \subset X$ is said to be **fragmented by ρ down to ε** if, for every nonempty subset $A \subset X_1$, there exists a nonempty relatively open subset $B \subset A$ such that $\rho-$diameter$(B) < \varepsilon$.

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The following notion introduced by Jayne and Rogers [JR] was one of the inspiration points for our investigations. It concerns the situations when the topology of a given space X is not metrizable but has a specific relation to some metric ρ.

Definition 1 ([JR]) *Let (X, τ) be a topological space and ρ some metric on it. The space (X, τ) (or the topology τ) is said to be* **fragmented by the metric** *ρ, if every nonempty subset of X is fragmented by ρ down to ε, for every $\varepsilon > 0$.*

This notion suggested itself as a convenient tool in the investigation of, (i) the geometry of Banach spaces; (ii) the study of the set of points where a given convex function is (Gateaux or Frechet) differentiable and (iii) the generic well-posedness of optimization problems. Unfortunately, the weak topology on a Banach space is rarely fragmented by the norm metric. Much more frequently though, the following notion which was introduced and studied by Jayne, Namioka and Rogers in [JNR1] - [JNR5] does occur.

Definition 2 *A space X (or its topology τ) is said to be* **sigma-fragmented by a metric** *ρ if, for every $\varepsilon > 0$, there exists a countable family $(X_i)_{i \geq 1}$ of subsets of X such that:*

i) $X = \bigcup_{i \geq 1} X_i$;

ii) every X_i, $i = 1, 2, 3, \ldots$, is fragmented by ρ down to ε.

In what follows, if a Banach space E is under consideration, then the words "E is sigma-fragmented by the norm" or "E is sigma-fragmented" will mean that the weak topology on E is sigma-fragmented by the norm metric. We will also say that a given space is "fragmentable (sigma-fragmentable)", if it is fragmented (sigma-fragmented) by some metric. The following theorem (see [KM2], [KM3]) shows that in a Banach space setting the two notions, fragmentability and sigma-fragmentability are closely related.

Theorem 1 *For a subset X of a Banach space E the next properties are equivalent:*
i) X admits a metric ρ which fragments the weak topology and **majorizes** *the norm topology (i.e. the topology generated by the metric ρ contains the norm topology);*
ii) X admits a metric ρ which fragments the weak topology and majorizes the weak topology;
iii) X is sigma-fragmented by the norm.

In [KM1], [KM2] the fragmentability of a general topological space (X, τ) by a metric which majorizes some other topology τ' on X was characterized by the existence of a winning strategy for the player Ω in a special "fragmenting" game described below in Section 2. A slightly weaker condition, the absence of a winning strategy for the other player Σ, characterizes the spaces we investigate in this paper. In this way, the distinction between the spaces considered in this paper and fragmentable spaces is made explicit. We also give several equivalent conditions that characterize these spaces. All of which are in terms of the existence of points of τ'-continuity for some τ-continuous or "τ-quasi-continuous" mappings.

Definition 3 *A mapping $g : Z \to X$ between topological spaces Z and X is said to be* **quasi-continuous at** z_0 *if, for every open subset $U \subset X$ with $g(z_0) \in U$, there exists some open set $V \subset Z$ such that:*

a) $z_0 \in \overline{V}$ (the closure of V in Z);

b) $g(V) := \bigcup\{g(z) : z \in V\} \subset U$.

The mapping g is called **quasi-continuous** *if it is quasi-continuous at each point of Z.*

For real-valued functions the notion of "quasi-continuity" was introduced by Kempisty in [Kem]. However the roots of this notion go back to V. Volterra (see Baire [Ba], p. 94-95).

Intuitively, one might expect that quasi-continuous mappings have many points of continuity; which indeed is often the case. Levine ([Le]) has shown that if X has a countable base, then every quasi-continuous map $g : Z \to X$ can only be discontinuous at the points of a first Baire category subset of Z. Bledsoe ([Bl]) proved a similar result for the case when X is a metric space. Results of this kind can be found in many articles (see for instance, the survey papers [Ptr2], [Ptr3] of Piotrowski). There are however quasi-continuous mappings that are nowhere continuous. Take $Z := (0, 1)$ with the usual topology, $X := (0, 1)$ with the Sorgenfrey topology and the identity mapping $g : Z \to X$. Then the map g is quasi-continuous but nowhere continuous.

In Section 2 we consider spaces with two topologies τ and τ'. Theorem 4 gives a game characterization (in terms of the absence of a winning strategy for the player Σ in the fragmenting game) of the spaces (X, τ, τ') with the property:

> *For every quasi-continuous mapping $f : Z \to (X, \tau)$ defined on a complete metric space Z there exists a point at which f is τ'-continuous.*

In fact, the set of such points of τ'-continuity is dense in Z. Moreover, it is of the second Baire category in every nonempty open subset of Z. Theorem 4 also shows that if the space (X, τ, τ') has the above property, then it has the same property with respect to quasi-continuous mappings defined on spaces belonging to the class of α-favorable spaces, which is much larger than the class of completely metrizable spaces. The considerations in Section 2 are very similar to those contained in [KKM] where the partial case $\tau = \tau'$ was treated. There is one exception however, which is contained in Theorem 5 and concerns the case when the topology τ' is metrizable. Under this restriction, each one of the equivalent conditions listed in Theorem 4 which refer to quasi-continuous mappings is equivalent to the following condition (which only involves continuous mappings):

> *Every continuous mapping $f : Z \to (X, \tau)$ from an α-favorable space Z into (X, τ) is τ'-continuous at the points of a dense G_δ-subset of Z.*

When X is a subset of a Banach space E, τ is the weak and τ' the norm topology, one additional phenomenon has place. In Section 3 we give several characterizations of the class $\mathcal{L}$ of Banach spaces E such that, for every quasi-continuous mapping $f : Z \to (E, \text{weak})$ defined on a complete metric space Z, there exists a dense set of points at which f is norm continuous. The phenomenon we have in mind is expressed by the following statement:

A Banach space E belongs to the class $\mathcal{L}$ if and only if every quasi-continuous mapping $f : Z \to (E, weak)$ from a complete metric space Z has a point at which f is weakly continuous (not necessarily norm continuous).

This follows from Theorem 6 which establishes that each of the equivalent assertions listed in Corollary 2 are equivalent to any of the statements listed in Corollary 3.

The class $\mathcal{L}$ is closely connected with a result of Namioka [Na1]. Namioka proved that every weakly continuous mapping $f : Z \to E$, where Z belongs to a large class of topological spaces (including all complete metric spaces), must be norm continuous at the points of a dense subset of its domain. Note that the set of points of norm continuity of any mapping $f : Z \to E$ is always a G_δ set in Z. It was expected that the result of Namioka would remain valid for any arbitrary Baire space Z. Talagrand [Ta1] however provided a counter-example to this expectation by exhibiting a weakly continuous nowhere norm continuous mapping defined on an α-favorable space. This situation suggested the investigation of the class $\mathcal{N}$ (the class $\mathcal{T}$) of Banach spaces E for which every weakly continuous mapping $f : Z \to E$ defined on a Baire space (on an α-favorable space) Z is norm continuous at the points of dense G_δ-subset of Z. Clearly, $\mathcal{N}$ and $\mathcal{L}$ are subclasses of $\mathcal{T}$. An example of Haydon [Ha] based on a tree of Todorčević shows that $\mathcal{N}$ and $\mathcal{T}$ are distinct classes of Banach spaces. On the other hand we prove in Section 3, Corollary 3, that the classes $\mathcal{T}$ and $\mathcal{L}$ coincide.

We also show (see Proposition 3) that $E = l^\infty$ and $E = l^\infty/c_0$ do not belong to $\mathcal{T}$. In both cases we explicitly describe how to construct a weakly continuous mapping $h : Z \to E$ defined on a completely regular α-favorable space Z which is nowhere norm continuous. This brings clarity to a concern expressed by Haydon (see [Ha], end of page 30, beginning of page 31).

We give similar results for Banach spaces of the type $C(T)$, where T is a compact space. We prove that, if every mapping $f : Z \to C(T)$ which is defined on a complete metric space Z and is quasi-continuous with respect to τ_p (the topology of pointwise convergence on T) has a point at which it is τ_p-continuous, then every such f has a dense set of points at which it is norm continuous. This is used in Section 4 to establish the existence of points of joint continuity of functions of two variables which have a property slightly weaker than separate continuity.

At the end of this introductory section we show how fragmentability of a Banach space (E, weak) by a metric majorizing the norm topology on E implies the existence of many points of norm continuity for a weakly quasi-continuous mapping defined on a Baire space. Note that, in view of Theorem 1, Banach spaces admitting such a fragmenting metric are precisely those Banach spaces that are sigma-fragmented by their norm.

Theorem 2 *Let Z be a topological space and (X, τ) be a topological space which is fragmented by some metric ρ. Suppose that $f : Z \to (X, \tau)$ is a quasi-continuous mapping. Then there exists a subset $C(f) \subset Z$ such that:*

i) $Z \setminus C(f)$ is of the first Baire category in Z (i.e. $C(f)$ is a **residual** *subset of Z);*

ii) at the points of $C(f)$ the mapping f is ρ-continuous.

In particular, if the topology generated by the metric ρ majorizes some topology τ' on the space X, then $f : Z \to X$ is τ'-continuous at every point of the set $C(f)$.

Proof. Consider, for every $n = 1, 2, \ldots$, the set $V_n := \bigcup\{V : V \text{open in} \, Z \, \text{and} \, \rho-$diameter$(f(V)) \leq n^{-1}\}$. The set V_n is open in Z. It is also dense in Z. Indeed, suppose W is a nonempty open subset of Z. Consider the nonempty set $A := f(W) \subset X$. By the fragmentability of X there is some nonempty relatively open subset $B := A \cap U = f(W) \cap U$, where U is τ-open in X, such that ρ-diameter$(B) \leq n^{-1}$. The quasi-continuity of f implies that there is some nonempty open $V \subset W$ with $f(V) \subset U \cap f(W) = B$. This shows that $\emptyset \neq V \subset V_n \cap W$. Hence, V_n is dense in Z. Obviously, at each point of $C(f) := \bigcap_{n\geq 1} V_n$, the map f is ρ-continuous. ∎

Some remarks are in order in connection with this theorem and the notions used in it. Every metrizable space X is, of course, fragmentable. There are however many non-metrizable spaces that are fragmentable (see the papers of Namioka [Na2] and Ribarska [Ri1], [Ri2] for further information concerning fragmentable spaces).

The notion of quasi-continuity is frequently used when establishing the existence of points of joint continuity of separately continuous real-valued functions of two variables (for the further development in this direction over the years see the papers of Martin [Mrt], Marcus [Mrc], Mibu [Mi], Namioka [Na1], Piotrowski [Ptr1]- [Ptr4], Troallic [Tro]). The general form of the notion of quasi-continuity (for mappings between general topological spaces) turned out to be instrumental in the proof that some semitopological groups are actually topological groups (see Bouziad [Bou1]-[Bou2]) and in the proof of some generalizations of Michael's selection theorem (see Giles and Bartlett [GB]). A lot of information concerning quasi-continuity of mappings may be found in the survey paper [Neu2] of Neubrunn.

The study of the class $\mathcal{L}$ is a natural continuation of the considerations in [KG], [GKMS] and [MG]. In [MG] the spaces from the class $\mathcal{L}$ were called "GGC spaces" (General Generic Continuity Spaces) in contrast to some related classes of spaces which were called "Generic continuity spaces" that were considered in [KG], [GKMS].

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2 The game $G(X, \tau, \tau')$ and the continuity of quasi-continuous mappings.

The main technical tool in this paper is the topological game $G(X, \tau, \tau')$ which we call the "fragmenting game" and play in the following way. Two players Σ and Ω select, one after the other, subsets of X. Ω starts the game by selecting the whole space X. Σ answers by choosing any nonempty subset A_1 of X and Ω goes on by taking a nonempty subset $B_1 \subset A_1$ which is relatively τ-open in A_1. After the first n moves of the game, Σ selects any nonempty subset A_n of the last move B_{n-1} of Ω and the latter answers by taking again a nonempty relatively τ-open subset B_n of the set A_n, just chosen by Σ. Acting in this way, the players produce a sequence of nonempty sets $A_1 \supset B_1 \supset A_2 \supset \ldots \supset A_n \supset B_n \supset \ldots$, which is called a **play** and will be denoted by $p := (A_i, B_i)_{i\geq 1}$ (there is no need to include in this notation the space X which is the initial obligatory move of Ω). The winning rule depends upon the topology τ'. The player Ω is said to have **won** the play $p := (A_i, B_i)_{i\geq 1}$, if the set $\bigcap_{n\geq 1} A_n$ is either empty or contains exactly one point x and for every τ'-open neighborhood U of x, there is some positive n with $B_n \subset U$. Otherwise the player Σ is said to have won the play.

A **partial play** is a finite sequence which consists of the first several moves $A_1 \supset B_1 \supset A_2 \supset ... \supset A_n$ (or $A_1 \supset B_1 \supset A_2 \supset ... \supset B_n$) of a play. A **strategy** ω for the player Ω is a mapping which assigns to each partial play $A_1 \supset B_1 \supset A_2 \supset ... \supset A_n$ some set B_n such that $A_1 \supset B_1 \supset A_2 \supset ... \supset A_n \supset B_n$ is again a partial play. A strategy σ for Σ is defined in a similar way. Sometimes we will denote the first choice A_1 under a strategy σ by $\sigma(X)$. A **σ-play (ω-play)** is a play in which Σ (Ω) selects his/her moves according to the strategy σ (ω). The strategy ω (σ) is said to be a **winning** one if every ω-play (σ-play) is won by Ω (Σ). The game $G(X, \tau, \tau')$ or the space X is called **Ω-favorable** (**Σ-favorable**), if there is a winning strategy for the player Ω (Σ). The game $G(X, \tau, \tau')$ (or the space X) is called **Σ-unfavorable**, if there does *not* exist a winning strategy for the player Σ.

This game has already been used to characterize the fragmentability of a topological space. The following statement was proved in [KM1], [KM2].

Theorem 3 *A topological space (X, τ) is fragmentable by a metric ρ which majorizes a topology τ' if and only if there exists a winning strategy for the player Ω in the game $G(X, \tau, \tau')$.*

When τ' is the trivial topology (consisting of the empty set and the whole space X), this yields a criterion for fragmentability of (X, τ). One derives from this theorem and Theorem 2 that if the game $G(X, \tau, \tau')$ is Ω-favorable, Z is a complete metric space and $f : Z \to (X, \tau)$ is a quasi-continuous map, then the set of points $C(f, \tau')$ where f is τ'-continuous contains a dense and G_δ-subset of Z.

For non-fragmentable spaces X one should not expect that the set $C(f, \tau')$ is always residual in Z. However density of this set can have place even without fragmentability of (X, τ) by a metric that majorizes τ'. As we shall see it suffices to know that the game $G(X, \tau, \tau')$ is unfavorable for the player Σ. To give a general formulation of our results we need the concept of α-favorability, which we introduce by means of a topological game which is the grandparent of many other topological games. This is the well known "Banach-Mazur game" (also called the "Choquet game").

Let Z be a topological space. The Banach-Mazur game $BM(Z)$ is played by two players α and β, who select alternatively nonempty open subsets of Z. It is β who starts the game by taking some nonempty open subset V_0 of Z. At the n-th move, $n \geq 1$, the player α takes a nonempty open subset $W_n \subset V_{n-1}$ and β answers by taking a nonempty open subset V_n of W_n. Using this way of selection, the players generate a sequence $(W_n, V_n)_{n=1}^{\infty}$ which is called a **play**. The player β is said to have **won** this play if $\bigcap_{n\geq 1} W_n = \emptyset$; otherwise this play is won by α. A **partial play** is a finite sequence which consists of the first several moves of a play. A **strategy** ζ for the player α is a mapping which assigns to each partial play $(V_0, W_1, V_1, W_2, V_2 ..., W_{n-1}, V_{n-1})$ some nonempty open subset W_n of V_{n-1}. A ζ-**play** is a play in which α selects his/her moves according to ζ. The strategy ζ is said to be a **winning** one if every ζ-play is won by α. A space Z is called α**-favorable** if there exists a winning strategy for α in $BM(Z)$. Also in this game it is possible to consider that the player α starts every play by always selecting the set $W_0 := Z$.

Let us recall that a space Z is called **Čech complete** if it is a G_δ-subset of some compact space. Z is said to be **almost Čech complete** if it contains a dense Čech complete subset. It is known that complete metric spaces are Čech complete and that every almost Čech complete space is α-favorable. Below we will also use the simple observation that for any α-favorable space Z and any subset H which is of the first Baire category in Z, there exists a strategy ζ for player α such that $\bigcap_{i\geq 0} W_i \neq \emptyset$ and $H \bigcap (\bigcap_{i\geq 0} W_i) = \emptyset$, whenever $(V_i, W_i)_{i\geq 0}$ is a ζ-play.

Theorem 4 *Let τ, τ' be two T_1 topologies on a set X. Suppose that for every τ'-open set U and every point $x \in U$ there exists a τ'-neighborhood V of x such that $\overline{V}^{\tau} \subset U$. Then the following conditions are equivalent:*

(i) The game $G(X, \tau, \tau')$ is Σ-unfavorable;

(ii) every quasi-continuous mapping $f : Z \to (X, \tau)$ from the complete metric space Z into (X, τ) has at least one point of τ'- continuity;

(iii) every quasi-continuous mapping $f : Z \to (X, \tau)$ from an α-favorable space Z into (X, τ) is τ'-continuous at the points of a subset which is second category in every nonempty open subset of Z.

Note that the absence of a winning strategy for the player Σ does not necessarily imply that Ω has a winning strategy, that is, the condition "the game $G(X, \tau, \tau')$ is Σ-unfavorable (or the space X is Σ-unfavorable)" is weaker than the condition "(X, τ) is fragmentable by a metric that majorizes τ'". As an example in this direction one could take the space $C(T)$ constructed by Haydon [Ha]. It will become clear after Corollary 3, that the game $G(C(T), \tau_p, \text{norm})$ is unfavorable for both players.

Proof of Theorem 4. We only outline the proof of this theorem here because it is very similar to the proof of the main result from [KKM]. We show that $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (i)$. The implication $(iii) \Rightarrow (ii)$ is obvious.

$(i) \Rightarrow (iii)$. Suppose X is Σ-unfavorable for $G(X, \tau, \tau')$ and $f : Z \to X$ is a τ-quasi-continuous mapping from the α-favorable space Z. Let H be any first Baire category subset of Z. There is some winning strategy ζ for the player α in $BM(Z)$ which "avoids" the set H, that is, $\bigcap_{i\geq 0} W_i \neq \emptyset$ and $H \bigcap (\bigcap_{i\geq 0} W_i) = \emptyset$ whenever $(V_i, W_i)_{i\geq 0}$ is a ζ-play. Take an open $V_0 \neq \emptyset$, $V_0 \subset Z$. We will show that f is continuous at some point of $V_0 \setminus H$. To do this we first construct a strategy σ for the player Σ in $G(X, \tau, \tau')$ and then use the fact that Σ does not win some σ-play. Define the first move of β in $BM(Z)$ to be V_0 and let $W_1 := \zeta(V_0)$ be the answer of α. Assign $A_1 := f(W_1)$ to be the first move in the strategy σ. Suppose that the answer of Ω in $G(X, \tau, \tau')$ is B_1, a nonempty relatively τ-open subset of A_1. Then the τ-quasi-continuity of f implies there exists some nonempty open subset V_1 of W_1, such that $f(V_1) \subset B_1$. Suppose the set V_1 is the next move of the player β in the game $BM(Z)$. The player α, of course, uses the strategy ζ to answer this move and selects the set $W_2 := \zeta(V_0, W_1, V_1)$. Then we define the second move of Σ in $G(X, \tau, \tau')$ to be $A_2 := \sigma(A_1, B_1) := f(W_2)$. Proceeding like this, we inductively construct a strategy σ. Together with each σ-play $(A_i, B_i)_{i\geq 1}$ in $G(X, \tau, \tau')$ we also construct a ζ-play $(W_i, V_i)_{i\geq 1}$ in $BM(Z)$ with $A_n := f(W_n)$ and $W_n := \zeta(V_0, W_1, V_1, ..., W_{n-1}, V_{n-1})$ for $n = 1, 2, \ldots$.

As ζ is a winning strategy for α, we have $\bigcap_{i\geq 1} W_i \neq \emptyset$. Therefore $\emptyset \neq f(\bigcap_{i\geq 1} W_i) \subset \bigcap_{i\geq 1} f(W_i) = \bigcap_{i\geq 1} A_i$. Since X is Σ-unfavorable, there is some σ-play $(A_i, B_i)_{i\geq 1}$ that is won by Ω; hence the nonempty set $\bigcap_{i\geq 1} A_i$ consists of just one point x such that for every τ'-neighborhood U of x there is some n with $A_n = f(W_n) \subset U$. This implies that $f(z) = x$ for every $z \in \bigcap_{i\geq 1} W_i \subset V_0 \setminus H$ and that f is continuous at each such z.

$(ii) \Rightarrow (i)$. Let σ be an arbitrary strategy for the player Σ in $G(X, \tau, \tau')$. We will show that it is not a winning one. Consider the space P of all σ-plays $p := (A_i, B_i)_{i\geq 1}$ endowed with the Baire metric d; that is, if $p := (A_i, B_i)_{i\geq 1} \in P$ and $p' := (A'_i, B'_i)_{i\geq 1} \in P$, then $d(p, p') := 0$ if $p = p'$ and $d(p, p') := 1/n$ where $n := \min\{k : B_k \neq B'_k\}$, if $p \neq p'$. Note that all the plays in P start with the same set $A_1 := \sigma(X)$, the first choice of the strategy σ. Also, if $A_i = A'_i$ and $B_i = B'_i$ for all $i \leq n$, then

$$A_{n+1} := \sigma(A_1, B_1, ..., A_n, B_n) = \sigma(A'_1, B'_1, ..., A'_n, B'_n) := A'_{n+1}.$$

In other words, if $p \neq p'$, then there is some n, such that $B_n \neq B'_n$, $A_i = A'_i$ for $i \leq n$ and $B_i = B'_i$ for $i < n$. It is easy to verify that (P, d) is a complete metric space.

Consider the (set-valued) mapping $F : P \to X$ defined by $F((A_i, B_i)_{i\geq 1}) := \bigcap_{i\geq 1} A_i$. If for some σ-play p we have $F(p) = \emptyset$, then the play p is won by Ω and there is nothing to prove. Therefore, without loss of generality, we may assume that F is nonempty-valued at every point of P. Let $f : P \to X$ be an arbitrary selection of the nonempty-valued map $F : P \to X$ (i.e. $f(p) \in F(p)$ for every $p \in P$). Next we will show that f is τ-quasi-continuous. Then, by property *(ii)*, f will be τ'-continuous at some point $p_0 \in P$. Finally we will show (see Proposition 1 below) that the play p_0 is won by Ω. This will show that σ is not a winning strategy and will complete the proof.

The next simple Lemma which is similar to Proposition 2.3 of [KO] plays an important role for our considerations.

Lemma 1 *Let the play $p_0 := (A_i, B_i)_{i\geq 1}$ be an element of the space P and let U a τ-open subset of X with $U \cap A_n \neq \emptyset$ for every $n = 1, 2, 3, \ldots$. Then there exists an open subset V in P such that:*

a) $p_0 \in \overline{V}$;

b) $F(V) := \bigcup\{F(p) : p \in V\} \subset U$.

Proof of the lemma. Let $p_0 := (A_i, B_i)_{i\geq 1}$ and U be as required in the formulation of the Lemma. Given a positive integer n, consider the nonempty set $B'_n := A_n \cap U$ (which is relatively τ-open in A_n and is a possible move of the player Ω). Denote by A'_{n+1} the set $\sigma(A_1, \ldots, B'_n)$ which is the answer of player Σ by means of the strategy σ. Let $p' \in P$ be some play in $G(X, \tau, \tau')$ which starts with the partial play $(A_1, \ldots, A'_{n+1})$. Clearly, $d(p_0, p') \leq n^{-1}$. Moreover, the d-ball $D(p_0, n^{-1}) := \{p : d(p_0, p) \leq n^{-1}\}$ contains the ball $D(p', (n+1)^{-1})$ and for every play p'' in the latter ball we have $F(p'') \subset B'_n \subset U$. Put V_n to be the interior of $D(p', (n+1)^{-1})$. Thus, for every integer $n > 0$, we found an open subset $V_n \subset D(p_0, n^{-1})$ such that $F(V_n) \subset U$. The set $V := \bigcup_{n\geq 1} V_n$ satisfies the requirements of *a)* and *b)*. ■

This Lemma immediately yields:

Corollary 1 *Every single-valued selection f of the set-valued mapping $F : P \to X$ defined above is τ-quasi-continuous.*

By *(ii)* f has a point of τ'-continuity. To complete the proof of the theorem we need the following statement.

Proposition 1 *Let f be an arbitrary single-valued selection of the set-valued mapping $F : P \to X$. If f is τ'-continuous at some point $p_0 \in P$, then the play $p_0 := (A_i, B_i)_{i\geq 1}$ is won by the player Ω in the game $G(X, \tau, \tau')$.*

Proof of the proposition. Let W be a τ'-open subset of X with $f(p_0) \in W$. Since f is τ'-continuous at $p_0 := (A_i, B_i)_{i\geq 1}$, there exists some open $V' \subset P$, $p_0 \in V'$, with $f(V') \subset W$. We will show that there is some integer $n > 0$ for which $A_n \subset \overline{W}^\tau$. In view of the relation between the topologies τ and τ' assumed in the formulation of the theorem, this will suffice to deduce that the play p_0 is won by the player Ω.

Suppose that the τ-open set $U := X \setminus \overline{W}^\tau$ intersects all sets A_n, $n = 1, 2, \ldots$. By the above Lemma 1, there is some open set $V \subset P$ such that $p_0 \in \overline{V}$ and $f(V) \subset F(V) \subset U$. In particular, $V \neq \emptyset$. Hence there is a point $p' \in V \cap V' \neq \emptyset$. For p' we have the contradiction: $f(p') \in U \cap W = \emptyset$. This shows that, for some $n > 0$, $A_n \subset \overline{W}^\tau$. ■

This completes the proofs of both Proposition 1 and Theorem 4. ■

Remark 1 The relation $A_n \subset \overline{W}^\tau$ implies that $F(D(p_0, n^{-1})) \subset \overline{W}^\tau$. Since W was an arbitrary τ'-open neighborhood of $f(p_0)$, we can derive from here that $F(p_0) = f(p_0)$ and that F is τ'-upper semi-continuous at p_0.

Theorem 5 *Under the hypotheses of Theorem 4 and the additional assumption that τ' is a metrizable topology the list of equivalent conditions in Theorem 4 can be extended by the following one:*

(iv) every continuous mapping $f : Z \to (X, \tau)$ from an α-favorable space Z into (X, τ) is τ'-continuous at the points of a dense G_δ-subset of Z.

Proof. We need a simple fact which is probably of independent interest as well.

Proposition 2 *Let $f : Z \to X$ be an open and quasi-continuous mapping defined on an α-favorable space Z. Then the image of f is an α-favorable space. In particular, for any quasi-continuous mapping $f : Z \to X$ defined on an α-favorable space Z the graph $G(f) := \{(z, x) \in Z \times X : x = f(z)\}$ of f is α-favorable.*

Proof of the Proposition. Let ζ be a winning strategy for the player α in the Banach-Mazur game $BM(Z)$ and let L_1 be the first move of the player β in the Banach-Mazur game played on $f(Z)$. By the quasi-continuity of f there exists a nonempty open subset $V_1 \subset Z$ such that $f(V_1) \subset L_1$. Consider V_1 as the first move of β in the game

$BM(Z)$ and denote by, $V_1' := \zeta(V_1)$ the answer of α according to the strategy ζ. We define a strategy η for α in $BM(f(Z))$ as follows:

$$L_1' := \eta(L_1) := f(V_1').$$

Proceeding inductively we construct a strategy η so that for every η-play $(L_1, L_1')_{i\geq 1}$ in $BM(f(Z))$ there corresponds a ζ-play $(V_i, V_i')_{i\geq 1}$ in $BM(Z)$ with the following properties that are fulfilled for every $i \geq 1$:

i) $f(V_i) \subset L_i$;

ii) $L_i' := \eta(L_1, L_1', \ldots, L_i) := f(V_i')$, where $V_i' := \zeta(V_1, V_1', \ldots, V_i)$.

Since ζ is a winning strategy there is a point $z_0 \in \bigcap_{i\geq 1} V_i'$ and so $f(z_0) \in \bigcap_{i\geq 1} L_i'$. This completes the first part of the proof. To deduce that $G(f)$ is α-favorable whenever f is quasi-continuous and Z is α-favorable we need only apply the previous result to the function $F : Z \to G(f)$ defined by, $F(z) := (z, f(z))$. ■

Note that in the previous Proposition if both X and Z are completely regular then so is $G(f)$.

We now return to the proof of the theorem. It is clear that condition *(iv)* is implied by condition *(iii)* of Theorem 4. We show now that *(iv)* implies *(iii)*. Denote by ρ some metric generating the topology τ'. Let $f : Z \to X$ be a τ-quasi-continuous mapping and Z an α-favorable space. Denote by π the standard projection of $Z \times (X, \tau)$ onto (X, τ). The restriction of π to the graph $G(f)$ of f is continuous and by *(iv)* and the above Proposition, will also be τ'-continuous at the points of a dense subset of $G(f)$. Consider some $\varepsilon > 0$ and some nonempty open subset $W \subset Z$. It suffices to show that there exists a nonempty open subset $V' \subset W$ such that the ρ-diameter of the set $f(V')$ is smaller than or equal to ε. Indeed, having this done, we could proceed as in the proof of Theorem 2. This would provide us with the dense and open sets $V_n := \bigcup\{V : V \text{open in} \, Z \, \text{and} \, \rho - \text{diameter}(f(V)) \leq n^{-1}\}$, $n = 1, 2, \ldots$, such that f is ρ-continuous at the points of the intersection $\bigcap_{n\geq 1} V_n$.

Since W is open, the set $W \times X$ is also open and intersects the α-favorable space $G(f)$. Therefore there exists some point $(z^*, f(z^*)) \in W \times X$ at which π is τ'-continuous. Then, for some open sets $V \subset Z$ and $U \subset X$ containing z^* and $f(z^*)$ respectively, we have $\rho - \text{diam}(\pi((V \times U) \bigcap G(f))) \leq \varepsilon$. By the quasi-continuity of f there is some nonempty open $V' \subset V$ such that $f(V') \subset U$. Clearly, $f(V') \subset \pi((V \times U) \bigcap G(f))$. ■

3 Norm continuity of quasi-continuous mappings into Banach spaces.

In this section we consider two particular cases of the general results from the preceeding section (Theorem 4 and Theorem 5). These are the cases when X is a subset of some Banach space E and either:

a) $\tau = \tau' =$ weak or,

b) $\tau =$ weak and $\tau' =$ norm.

In case a) Theorem 4 immediately yields.

Corollary 2 *The following properties of a subset X of a Banach space E are equivalent:*
(i) $G(X, weak, weak)$ is Σ-unfavorable;
(ii) every quasi-continuous mapping $f : Z \to (X, weak)$ from a complete metric space Z is weakly continuous at some point of Z;
(iii) every quasi-continuous mapping $f : Z \to (X, weak)$ from an α-favorable space Z is weakly continuous at the points of a subset of Z which is second category in every nonempty open subset of Z.

In the case b) where τ' is a metrizable topology, the set of τ'-continuity points is always a G_δ set. Hence, from Theorem 4 and Theorem 5, we have the following result.

Corollary 3 *The following properties of a subset X of a Banach space E are equivalent:*
(i) $G(X, weak, norm)$ is Σ-unfavorable;
(ii) every quasi-continuous mapping $f : Z \to (X, weak)$ from a complete metric space Z is norm continuous at some point of Z;
(iii) (E belongs to $\mathcal{L}$) every quasi-continuous mapping $f : Z \to (X, weak)$ from a complete metric space Z is norm continuous at the points of some dense and G_δ subset of Z;
(iv) every quasi-continuous mapping $f : Z \to (X, weak)$ from an α-favorable space Z is norm continuous at the points of a dense and G_δ subset of Z;
(v) (E belongs to $\mathcal{T}$) every continuous mapping $f : Z \to (X, weak)$ from a completely regular α-favorable space Z is norm continuous at the points of a dense and G_δ subset of Z. In particular, the classes $\mathcal{L}$ and $\mathcal{T}$ coincide.

Note that the equivalence between *(i)* and *(iii)* in the last corollary was proved in Theorem 5.2 of [MG].

The main aim of this section is to present a phenomenon which is specific to Banach spaces. It implies that every one of the equivalent conditions in Corollary 2 is equivalent to any of the conditions in Corollary 3. Thus the fact that a given Banach space belongs to the class $\mathcal{L}$ can be expressed in many different ways.

Theorem 6 *Let X be a subset of a Banach space E. The following conditions are equivalent:*
(a) there is a winning strategy for the player Σ in $G(X, weak, weak)$;
(b) there is a winning strategy for the player Σ in $G(X, weak, norm)$;
(c) there is a strategy σ' for the player Σ such that, for every σ'- play $(A'_i, B'_i)_i$, the set $\bigcap_{i\geq 1} A'_i \neq \emptyset$ and there is at least one sequence $(x_i)_{i\geq 1}$ with $x_{i+1} \in A'_i$, that has no cluster points in $(E, weak)$.
In particular, conditions (i) from Corollary 2 and Corollary 3 are equivalent to each other.

Proof. The implications $(c) \Rightarrow (a) \Rightarrow (b)$ are evident. We prove now that $(b) \Rightarrow (c)$. This will be done by a construction already used in [KM3] to show that the player Ω has a winning strategy in $G(X$, weak, weak) if and only if he/she has a winning strategy in the game $G(X$, weak, norm). Before that the same idea was exploited by Christensen [Chr] to show that every weakly continuous mappings defined on a "good" space must be norm continuous at many points.

Suppose σ is the winning strategy for Σ in the game $G(X, \text{weak, norm})$. We will construct a strategy σ' with the property described in (c). For technical reasons, we need the first choice under σ' to be a bounded subset of E. The following statement allows us to do so.

Lemma 2 *If there is a winning strategy σ for the player Σ in the game $G(X, weak, norm)$, then there is a winning strategy σ^* for the same player (in the same game) such that $A_1^* := \sigma^*(X)$ is a bounded subset.*

Proof of the lemma. Assume the set $\sigma(X) := A_1 \subset X$ is the first choice of the player Σ under the strategy σ. If A_1 is a subset of the closed unit ball of B of E, there is nothing to prove. If this is not the case we consider the relatively open (and nonempty) set $B_1 := A_1 \bigcap \{x : \|x\| > 1\}$ and the set $A_2 := \sigma(A_1, B_1)$. If A_2 is a subset of $2B$, we set $A_1^* := \sigma^*(X) := A_2$. In this case, in the next steps we can apply the winning strategy σ:

$$\sigma^*(A_1^*, B_1^*, \ldots, A_k^*, B_k^*) := \sigma(A_1, B_1, A_1^*, B_1^*, \ldots, A_k^*, B_k^*).$$

In this way we define a winning strategy σ^*. If A_2 is not a subset of $2B$, we consider the nonempty sets $B_2 := A_2 \bigcap \{x : \|x\| > 2\}$ and $A_3 := \sigma(A_1, B_1, A_2, B_2)$. If $A_3 \subset 3B$, we put $A_1^* := \sigma^*(X) := A_3$ and play further according to the winning strategy σ. Continuing in this way we must arrive at some k for which $A_k \subset kB$. Otherwise a σ-play $p = (A_i, B_i)_{\imath \geq 1}$ will appear for which

$$B_i := A_i \bigcap \{x : \|x\| > i\} \neq \emptyset$$

for every $i \geq 1$. Such a play p would be won by Ω (the intersection of the elements of the play would be empty) which is a contradiction. Let $m > 0$ be the first integer for which $A_m \subset mB$. We put $A_1^* := \sigma^*(X) := A_m$ and define the strategy σ^* as follows:

$$\sigma^*(A_1^*, B_1^*, \ldots, A_k^*, B_k^*) := \sigma(A_1, B_1, \ldots, B_{m-1}, A_1^*, B_1^*, \ldots, A_k^*, B_k^*).$$

This completes the proof of the lemma. ∎

Without loss of generality we may assume that $A_1 \subset B$. Take some $x_1 \in A_1$ and put $d_1 := \inf\{t > 0 : \ tB \supset A_1\}$. We have $d_1 > 0$, as otherwise A_1 would be a singleton and Ω would win every play in the game $G(X, \text{weak, norm})$. The set $A_1 \setminus \frac{1}{2}d_1 B$ is nonempty and relatively open in A_1. Take some open U such that $A_1' := U \bigcap A_1 \neq \emptyset$ and $\overline{U} \bigcap (x_1 + \frac{1}{2}d_1 B) = \emptyset$ (here, as everywhere in this proof, we denote by $\overline{C}$ the closure in (E, weak) of the set C). Define $\sigma'(X) := A_1'$. Note that A_1' is nonempty and relatively open in A_1 and $\overline{A_1'} \bigcap (x_1 + \frac{1}{2}d_1 B) = \emptyset$. Let the answer of Ω to this move be some relatively open subset B_1 of A_1' (and therefore of A_1). Then (A_1', B_1) is a partial σ'-play and (A_1, B_1) is a partial σ-play. Suppose that, in the course of defining the strategy σ', we have constructed the partial σ-play $p_n := (A_i, B_i)_{i=1}^n$, the partial σ'-play $p_n' := (A_i', B_i)_{i=1}^n$, the points $(x_i)_{i=1}^n$ and the numbers $(d_i)_{i=1}^n$ so that, for every $i = 1, 2, 3 \ldots, n$,
i) A_i' is a relatively open subset of A_i;
ii) $x_i \in A_i$;

iii) $d_i := \inf\{t > 0 : \ co(x_1, \dots, x_i) \ + \ tB \supset A_i\} > 0$, where $co(x_1, \dots, x_i)$ stands for the convex hull of the set $\{x_1, \dots, x_i\}$;
iv) The closure $\overline{A'_i}$ of A'_i in (E, weak) does not intersect the set $co(x_1, \dots, x_i) + \frac{i}{i+1} d_i B$;
v) $\| \cdot \| - \mathrm{diam}(A'_i) \le 2(d_i + \frac{1}{i+1})$.

Let $A_{n+1} := \sigma(p_n)$ be the next choice of Σ in the game $G(X, \text{weak, norm})$. Take some $x_{n+1} \in A_{n+1}$ and put

$$d_{n+1} := \inf\{t > 0 : co(x_1, \dots, x_{n+1}) + tB \supset A_{n+1}\}.$$

We must have $d_{n+1} > 0$, since otherwise the set A_{n+1} would be a subset of a finite-dimensional linear space in which the weak and the norm topology coincide and Ω would have an obvious winning strategy. Consider the nonempty set

$$A_{n+1} \setminus (co(x_1, \dots, x_{n+1}) + \frac{n+1}{n+2} d_{n+1} B)$$

and take some nonempty relatively open subset A of it such that

$$\overline{A} \bigcap (co(x_1, \dots, x_{n+1}) + \frac{n+1}{n+2} d_{n+1} B) = \emptyset.$$

Clearly, A is a relatively weakly open subset of A_{n+1}. Now there is a minimal (with respect to cardinality) finite set M such that

$$co(x_1, \dots, x_{n+1}) \subset (M + \frac{1}{n+2} B).$$

Since $A \subset A_{n+1} \subset co(x_1, \dots, x_{n+1}) + d_{n+1} B$, we have $A \subset M + (d_{n+1} + \frac{1}{n+2})B$. Then, for some $m_0 \in M$, the set

$$A'_{n+1} := A \setminus [(M \setminus \{m_0\}) + (d_{n+1} + \frac{1}{n+2})B] \neq \emptyset.$$

Since $A'_{n+1} \subset m_0 + (d_{n+1} + \frac{1}{n+2})B$, we have $\| \cdot \| - \mathrm{diam}(A'_{n+1}) \le 2(d_{n+1} + \frac{1}{n+2})$.

Define the move of Σ under σ' to be $\sigma'(p'_n) := A'_{n+1}$. By the construction it is a relatively open subset of A_{n+1}. Let B_{n+1} be a relatively open subset of A'_{n+1}. It is relatively open in A_{n+1} as well. Thus we constructed the partial σ-play $p_{n+1} := (A_i, B_i)_{i=1}^{n+1}$ and the partial σ'-play $p'_{n+1} := (A'_i, B_i)_{i=1}^{n+1}$ satisfying the conditions *i) – v)*. This considered as an inductive step, completes the construction of the strategy σ'. Note that the sets B_i in both partial plays are the same. Hence $A_{i+1} \subset B_i \subset A'_i$ and $x_{i+1} \in A'_i$. The sequence $(d_i)_{i \ge 1}$ of non-negative numbers is non-increasing. Put $d_\infty := \lim_{n \to \infty} d_n$. As $(A_i, B_i)_{i \ge 1}$ is a σ-play and σ is a winning strategy for Σ in $G(X, \text{weak, norm})$, the intersection $\bigcap_{i \ge 1} A_i = \bigcap_{i \ge 1} A'_i$ is nonempty and $\lim_{n \to \infty} \| \cdot \| - \mathrm{diam}(A_{n+1}) = \lim_{n \to \infty} \| \cdot \| - \mathrm{diam}(A'_{n+1}) > 0$. Hence, by property (v), we have $d_\infty > 0$. We will show that the sequence $(x_i)_{i \ge 1}$ has no weak cluster points in E, thus proving (c). Assume that it has a weak cluster point x_∞; it necessarily belongs to $\bigcap_{i \ge 1} \overline{A'_i}$. Since the sequence (x_i) is in a Banach space, the point x_∞ must belong to the norm closure of the convex set $\bigcup_{i \ge 1} co(x_1, \dots, x_i) = co(\bigcup_{i \ge 1} \{x_i\})$. Property *iv)* however implies that the latter set does not intersect the norm ball of radius $\frac{1}{2} d_\infty$ centered at x_∞. This contradiction completes the proof of the theorem. ∎

Corollary 4 *The class $\mathcal{L} = \mathcal{T}$ is preserved by weak-to-weak homeomorphisms. I.e. if E_1 and E_2 are Banach spaces such that E_1 belongs to $\mathcal{L}$ and $(E_1, weak)$ is homeomorph to $(E_2, weak)$, then E_2 also belongs to $\mathcal{L}$.*

Proof. This is so because condition (a) in Theorem 6 characterizes the class $\mathcal{L} = \mathcal{T}$ solely in terms of the weak topology. ■

Taken together, Theorem 1, Theorem 3 and Theorem 6 show that the two games $G(X, \text{weak}, \text{norm})$ and $G(X, \text{weak}, \text{weak})$ are equivalent, that is, each player has a winning strategy in one of the games if, and only if, she/he has a winning strategy in the other game.

Condition c) in the last theorem suggests that we consider another game, denoted $G^*(X, \tau, \tau')$ which is similar to the game $G(X, \tau, \tau')$. The moves of the players Ω and Σ in $G^*(X, \tau, \tau')$ are the same as in the game $G(X, \tau, \tau')$. Only the winning rule is different. By definition, the player Ω wins the play $p := (A_i, B_i)_{i\geq 1}$ in $G^*(X, \tau, \tau')$ if either $\bigcap_i A_i = \emptyset$ or (when this intersection is nonempty) every sequence $(x_i)_{i\geq 1}$ with $x_i \in A_i$, $i = 1, 2, \ldots$, has a τ'-cluster point. Clearly, some subset X of a Banach space E satisfies any of the equivalent conditions in Corollary 2 and Corollary 3 if, and only if, X is unfavorable for Σ in the game $G^*(X, \text{weak}, \text{weak})$. Also, it follows from Theorem 1.3 and Theorem 2.1 of [KM3] that the player Ω has a winning strategy in $G^*(X, \text{weak}, \text{weak})$ if, and only if, X is fragmentable by a metric that majorizes the norm topology. In view of Theorem 1, the latter happens if and only if (X, weak) is sigma-fragmentable by the norm.

In [Ha] (see bottom of page 30 and the beginning of page 31) Haydon comments that it is not clear if there exists a weakly continuous mapping $f : Z \to l^\infty$ defined on an α-favorable space which is nowhere norm continuous. It was conjectured that this "... may conceivably depend upon additional set-theoretic assumptions". The next statement clarifies this situation.

Proposition 3 *Neither of the Banach spaces l^∞ nor l^∞/c_0 belong to $\mathcal{L}$. Hence there exist weakly continuous functions defined on completely regular α-favorable spaces that map into l^∞ and l^∞/c_0 that are nowhere norm continuous.*

Proof. Let first $E = l^\infty$. In Proposition 5.1 from [KM3] it is proved that there exists a strategy σ for the player Σ in the game $G(E, \text{weak}, \text{norm})$ such that, for every play $p := (A_i, B_i)_{i\geq 1}$, the set $\bigcap_{i\geq 1} A_i$ is nonempty and $\lim_i(\text{norm-diameter} A_i) > 0$. Corollary 3 shows that there exists a weakly continuous mapping $h : Z \to E$ defined on a complete metric space which is nowhere norm continuous. This completes the proof. We would like however to describe this mapping h. Consider the complete metric space P of all σ-plays $p := (A_i, B_i)_{i\geq 1}$ and the set-valued mapping $F : P \to E$ defined by $F((A_i, B_i)_{i\geq 1}) := \bigcap_{i\geq 1} A_i$. The properties of σ imply the set $F(p)$ is nonempty for every $p \in P$. Let $f : Z \to E$ be some single-valued selection of F. Apply the proof of Theorem 4 (especially the part where the implication $(ii) \Rightarrow (i)$ was established) to the case when $X = E$, $\tau = \text{weak}$ and $\tau' = \text{norm}$. By Corollary 1, f is a weakly quasi-continuous mapping defined on the complete metric space Z. According to Proposition 1 and the properties of σ the mapping f is nowhere norm continuous. Consider the projection $\pi : G(f) \to (E, \text{weak})$ from the graph $G(f)$ of f into (E, weak). From Proposition 2 we know that $G(f)$ is completely regular and α-favorable. The proof of Theorem 5 reveals that there exists some open subset L of $G(f)$ at the points of which π is not norm continuous (otherwise f would be norm continuous at many points). Note that L is also α-favorable. Now we can put $h := \pi$, $Z := L$. Clearly, $h : Z \to E$ is a weakly continuous nowhere norm continuous mapping defined in a completely regular α-favorable space. This completes the proof for the case $E = l^\infty$.

The case $E = l^\infty/c_0$ is very similar. The only difference is that instead of Proposition 5.1 from [KM3] we apply Theorem 2.3 from [KM1]. The latter says that there exists a strategy σ for the player Σ in the game $G(E, \text{weak}, \text{norm})$ such that, for every play $p := (A_i, B_i)_{i\geq 1}$, the set $\bigcap_{i\geq 1} A_i$ has more than one point. ∎

4 Continuity of quasi-separately continuous functions of two variables

Let T be a compact space and $C(T)$ the space of continuous real-valued functions on T. Denote by τ_p, the topology of pointwise convergence on T and by "norm" the topology generated by the sup-norm. The following statements have place.

Theorem 7 *Let X be a subset of a space $C(T)$, for some compact space T. Then the following conditions are equivalent:*
(a) there is a winning strategy for the player Σ in $G(X, \tau_p, \tau_p)$;
(b) there is a winning strategy for the player Σ in $G(X, \tau_p, norm)$;
(c) there exists a strategy σ' for the player Σ such that, for every σ'-play $(A'_i, B'_i)_i$, the set $\bigcap_{i\geq 1} A'_i \neq \emptyset$ and there is some sequence $(x_i)_{i\geq 1}$ with $x_{i+1} \in A'_i$ that has no τ_p-cluster points in $C(T)$.

Proof. Clearly, $(c) \Rightarrow (a) \Rightarrow (b)$. The proof of $(b) \Rightarrow (c)$ coincides with the proof of the same implication in Theorem 6 up to the following single change. If x_∞ is any τ_p-cluster point of the sequence $(x_i)_{i\geq 1}$ then there is a subsequence $(x_{i_k})_{k\geq 1}$ which τ_p-converges to x_∞. This follows from the fact that, (i) $(x_i)_{i\geq 1}$ is a relatively countably τ_p-compact subset of B and (ii) $C(T)$ with the pointwise topology is angelic. The Lebesgue dominated convergence theorem then shows that $(x_{i_k})_{k\geq 1}$ converges weakly to x_∞. Hence x_∞ must belong to the norm closure of the convex set $co(\bigcup_{k\geq 1}\{x_{i_k}\})$ which contradicts property iv). This completes the proof of the theorem. ∎

In [KM3] it was noted that the games $G(X, \tau_p, \tau_p)$ and $G(X, \tau_p, \text{norm})$ are simultaneously favorable (or unfavorable) for the player Ω. The above theorem shows that the two games are equivalent. The example of Haydon [Ha], mentioned in the Introduction, is a space which is unfavorable for both players in these games.

We formulate here some assertions concerning the space $(C(T), \tau_p)$ which are in the style of the results from the previous section.

Corollary 5 *The following properties of a subset X of a $C(T)$ space are equivalent:*
(i) $G(X, \tau_p, \tau_p)$ is Σ-unfavorable;

(ii) every quasi-continuous mapping $f : Z \to (X, \tau_p)$ from a complete metric space Z is τ_p-continuous at some point of Z;
(iii) every quasi-continuous mapping $f : Z \to (X, \tau_p)$ from an α-favorable space Z is τ_p-continuous at the points of a subset which is second category in every nonempty open subset of Z.

Corollary 6 *The following properties of a subset X of a $C(T)$ space are equivalent:*
(i) $G(X, \tau_p, norm)$ is Σ-unfavorable;
(ii) every quasi-continuous mapping $f : Z \to (X, \tau_p)$ from a complete metric space Z is norm continuous at some point of Z;
(iii) every quasi-continuous mapping $f : Z \to (X, \tau_p)$ from an α-favorable space Z is norm continuous at the points of a dense (and G_δ) subset of Z;
(iv) every continuous mapping $f : Z \to (X, \tau_p)$ from a completely regular α-favorable space Z is norm continuous at the points of a dense (and G_δ) subset of Z.

Corollary 7 *i) All the conditions listed in the above two corollaries are equivalent to each other.*

ii) Let X_1, X_2 be subsets of $C(T_1)$, $C(T_2)$ correspondingly. If (X_1, τ_p) is homeomorph to (X_2, τ_p) and X_1 has some of the equivalent properties listed in the last two corollaries, then X_2 also has these properties.

Proof. Follows immediately from Theorem 7. ■

Let Z be a topological space, T a compact Hausdorff space and $f(z,t)$ a real-valued function defined on $Z \times T$. The function f is said to be **separately continuous** if for every $z_0 \in Z$ and every $t_0 \in T$ the functions $f(z_0, t)$ and $f(z, t_0)$ are continuous in T and Z respectively. Under rather mild restrictions imposed on the spaces Z and T it was established that for every separately continuous function $f(z,t)$ there exists a dense G_δ subset A of Z such that f is continuous at every point of $A \times T$ (see [Na1], [Bou1], [Ta1], [Ta2], [De]).

We establish here similar results for functions f which satisfy a requirement slightly weaker than separate continuity.

Definition 4 *We call a real-valued function $f(z,t)$* **quasi-separately continuous at** *(z_0, t_0) if $f(z_0, t)$ is continuous in T and for every finite set $K \subset T$ and every $\varepsilon > 0$ there exists some open $V \subset Z$ such that $z_0 \in \overline{V}$ and $|f(z,t) - f(z_0,t)| < \varepsilon$ whenever $z \in V$ and $t \in K$. The function f is called* **quasi-separately continuous** *if it is quasi-separately continuous at every point of $Z \times T$.*

Clearly, every separately continuous function f is quasi-separately continuous as well. Simple examples (in which T is a singleton) show that the two notions do not coincide. The terminology we use becomes natural if we adopt another point of view and consider the function $f(z,t)$ as a mapping from Z into the space R^T of all real-valued functions on T. This mapping (denoted by, $f : Z \to R^T$) puts into correspondence to every $z_0 \in Z$ the function $f(z_0)(t) := f(z_0, t)$. The function $f(z,t)$ is separately continuous, if and only if

$f(z) \in C(T)$ for every $z \in Z$ and the mapping $f : Z \to (C(T), \tau_p)$ is continuous. It is easy to see that $f(z,t)$ is separately quasi-continuous if and only if the map $f : Z \to (C(T), \tau_p)$ is well defined and quasi-continuous.

On the other hand, $f(z,t)$ is continuous at (z_0, t), for all $t \in T$, exactly when $f : Z \to C(T)$ is continuous at z_0 with respect to the norm in $C(T)$. Therefore the problem we discuss now is, in fact, identical with what we were studying above: characterize the situation when every quasi-continuous mapping $f : Z \to (C(T), \tau_p)$ defined on a complete metric space Z has a dense G_δ subset of points of norm continuity. The next statement presents several equivalent characterizations. They are either in the language of quasi-separately continuous or just separately continuous functions.

Theorem 8 *Let T be a compact space and $C(T)$ the space of continuous functions on it. The following properties are equivalent:*
(i) $G(C(T), \tau_p, \tau_p)$ is Σ-unfavorable;
(ii)(equivalent to condition (ii) from Corollary 5) for every quasi-separately continuous function $f : Z \times T \to R$, where Z is a complete metric space, there is a point $z_0 \in Z$ such that, for every fixed $t_0 \in T$ the function $f(z, t_0)$ is continuous at z_0;
(iii)(equivalent to condition (ii) from Corollary 6) for every quasi-separately continuous function $f : Z \times T \to R$, where Z is a complete metric space, there exists a point $z_0 \in Z$ such that f is continuous at each point of the set $\{z_0\} \times T$;
(iv)(equivalent to condition (iii) from Corollary 6) for every quasi-separately continuous function $f : Z \times T \to R$, where Z is an α-favorable space, there exists a dense G_δ subset $A \subset Z$ such that f is continuous at each point of the set $A \times T$;
(v)(equivalent to condition (iv) from Corollary 6) for every separately continuous function $f : Z \times T \to R$, where Z is a completely regular α-favorable space, there exists a dense G_δ subset $A \subset Z$ such that f is continuous at each point of the set $A \times T$;
(vi) $G(C(T), \tau_p, norm)$ is Σ-unfavorable.

Proof. Follows immediately from the Remark after Corollary 6. ∎

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