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The Relationship between Goldstine's Theorem and the Convex Point of Continuity Property

WARREN B. MOORS

Department of Mathematics and Statistics, The University of Auckland, Private Bag 92019, Auckland, New Zealand

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Goldstine's Theorem says that the natural embedding of the closed unit ball $B(X)$ of a Banach space X is weak* dense in the second dual ball $B(X^{**})$. In this paper we characterise, in terms of the geometry of $B(X)$, when the natural embedding of $B(X)$ into $B(X^{**})$ is not only weak* dense, but also residual. Using this characterisation, we show that a Banach space X has the convex point of continuity property, if and only if, for each equivalent norm ball $B(X)$, the natural embedding of $B(X)$ into $B(X^{**})$ is residual with respect to the weak* topology. We also show that a Banach space X has the Radon–Nikodym property if and only if, for each equivalent norm ball $B(X)$, the set of linear functionals in X^* which attain their norm on $B(X)$ is residual in X^*.

INTRODUCTION

In Section one we recall some well-known conditions which are sufficient for a set-valued mapping to be norm continuous on a dense subset of its domain.

In the remainder of this paper we consider some simple applications of these results to the geometry of Banach spaces.

We begin by first considering only separable Banach spaces.

For this class of Banach spaces we provide some simple, but elegant topological characterisations of the following properties.

(1) The point of continuity property and the convex point of continuity property;

(2) the weak* point of continuity property and the weak* convex point of continuity property.

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In Section three we extend the results given in Section two to non-separable Banach spaces.

Section four is concerned with the solution of the following conjecture, which was brought to the attention of the author by P. S. Kenderov.

A Banach space X has the Radon–Nikodym property if and only if, for each equivalent norm ball $B(X)$ on X the set of functionals in X^* which attain their norm on $B(X)$ is residual in X^*. This result is an advance on the following theorem, which is due to Bourgain [2].

A Banach space X has the Radon–Nikodym property if and only if, for each closed, bounded, convex subset C of X the set of functionals which attain their maximum value on C is residual in X^*.

Finally, Section five is concerned with expanding the characterisations given in Section three, using some new continuity properties introduced in [7, 10].

Notation. All the Banach spaces considered in this paper will be over the real numbers.

For a Banach space $(X, \|\cdot\|)$, we denote by:

X^*	the dual of X;
X^{**}	the second dual of X;
$B(X)$	the closed unit ball in X, with respect to a given norm;
$S(X)$	the unit sphere in X, with respect to a given norm;
$\hat{X}$	The natural embedding of X into X^{**}.

For a non-empty subset E of X we denote by:

$\operatorname{co} E$	the convex hull of E;
$\overline{\operatorname{co}}\, E$	the closed convex hull of E;
$\overline{E}$	the norm closure of E;
$\overline{E}^{w^*}$	the weak* closure of E.

1. Some Continuity Results

In this first section, we investigate two continuity results which have useful applications in the geometry of Banach spaces.

The first one is due to I. Namioka [11], and relates weak continuity to norm continuity for single-valued mappings from "nice" Baire spaces into Banach spaces.

The second result which has appeared in many different forms over the last twenty years, relates the separability of the range of a set-valued mapping to its norm continuity. We provide a simple self-contained proof of the form of this result that we require.

A set-valued mapping Φ from a topological space A into subsets of a topological space X is said to be *upper semi-continuous* at t in A if, given an open set W containing $\Phi(t)$ there exists an open neighbourhood U of t such that $\Phi(U)$ is contained in W, where $\Phi(U) \equiv \bigcup\{\Phi(t) : t \in U\}$. We call such a mapping an *usco* if for each $t \in A$, $\Phi(t)$ is non-empty and compact; when X is a linear topological space we call such a mapping a *cusco* if for each $t \in A$, $\Phi(t)$ is non-empty, convex, and compact.

In order to state our first theorem we will need the following definition.

Let $\mathscr{A}$ be an open covering of a topological space X. Then a subset S of X is said to be *$\mathscr{A}$-small* if S is contained in a member of $\mathscr{A}$.

A topological space X is said to be *strongly countably complete* if there is a sequence $\{\mathscr{A}_n : n \in \mathbb{N}\}$ of open coverings of X such that any sequence $\{F_n : n \in \mathbb{N}\}$ of non-empty closed subsets of X has $\bigcap\{F_n : n \in \mathbb{N}\} \neq \varnothing$ provided that $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}$ and each F_n is $\mathscr{A}_n$-small.

THEOREM 1.1. *Let A be a strongly countably complete regular space and let X be a Banach space. If $f : A \to (X, \text{weak})$ is a continuous map, then there is a dense G_δ subset G of A such that at each point of G, f is norm continuous.*

THEOREM 1.2. *Let A be a regular compact topological space; G be a dense and G_δ subset of A, and let f be a weak* continuous function from A into the dual of a Banach space X. If the restriction of f to G is weak continuous on G, then f is norm continuous on a dense G_δ subset of A.*

Proof. We begin by proving that G with the relative topology is a strongly countably complete regular topological space.

To this end, let $G \equiv \bigcap\{O_n : n \in \mathbb{N}\}$, where each O_n is a dense open subset of A.

For each $k \in \mathbb{N}$ and $x \in G$ choose an open neighbourhood $U_k(x)$ of x such that

$$\overline{U_k(x)} \subseteq O_k.$$

Let $\mathscr{A}_k \equiv \{U_k(x) \cap G : x \in G\}$. We now claim that a decreasing sequence $\{F_n : n \in \mathbb{N}\}$ of non-empty closed subsets of G has $\bigcap\{F_n : n \in \mathbb{N}\} \neq \varnothing$ provided that each F_n is $\mathscr{A}_n$-small; that is, provided that for each $n \in \mathbb{N}$ there exists an $x_n \in G$ such that $F_n \subseteq U_n(x_n)$. To see this, let $F_n \equiv F_n' \cap G$, where each F_n' is a closed subset of A.

Now,

$$\begin{aligned}\bigcap \{F_n : n \in \mathbb{N}\} &= \bigcap \{F_n' \cap G : n \in \mathbb{N}\} \\ &= \bigcap \{F_n' : n \in \mathbb{N}\} \cap \bigcap \{\overline{U_n(x_n)} : n \in \mathbb{N}\} \\ &= \bigcap \{F_n' \cap \overline{U_n(x_n)} : n \in \mathbb{N}\}.\end{aligned}$$

But $\bigcap\{F'_n \cap \overline{U_n(x_n)} : n \in \mathbb{N}\} \neq \emptyset$ by the compactness of A. Therefore $\bigcap\{F_n : n \in \mathbb{N}\}$ is non-empty. Hence G with the relative topology is strongly countably complete.

So from Theorem 1.1 there is a dense G_δ subset G_1 of G on which the restriction of f to G is norm continuous.

Next, we show that f is norm continuous wherever g, the restriction of f to G, is norm continuous. Let $t \in G$ and suppose g is norm continuous at t. Then given $\varepsilon > 0$ we can find an open neighbourhood U of t such that $g(U \cap G) \subseteq (g(t) + \varepsilon B(X^*))$. We now claim that $f(U) \subseteq (g(t) + \varepsilon B(X^*)) = (f(t) + \varepsilon B(X^*))$. Suppose this is not the case. Then there exists an $x \in U$ such that $f(x) \notin (f(t) + \varepsilon B(X^*))$. Now, since f is weak* continuous at x there exists a non-empty open subset V of U such that $f(V) \cap (f(t) + \varepsilon B(X^*)) \neq \emptyset$. However, for any $y \in V \cap G$ $f(y) = g(y) \in (g(t) + \varepsilon B(X^*))$ which is a contradiction; therefore we must have that f is norm continuous at t.

The result now follows by observing that G_1 is in fact a dense G_δ subset of A.

A weak* cusco mapping from a topological space A into subsets of the dual of a Banach space X is said to be *minimal* if its graph does not contain the graph of any other weak* cusco with the same domain.

The following well-known property of minimal weak* cuscos is given in [6, Lemma 2.5].

PROPOSITION 1.3. *A minimal weak* cusco Φ from a topological space A into subsets of the dual of a Banach space X has the property that, for each non-empty open subset V of A and weak* closed and convex subset K of X^*, if $\Phi(V) \not\subseteq K$ then there exists a non-empty open subset of V' of V such that $\Phi(V') \cap K = \emptyset$.*

PROPOSITION 1.4. *Let Φ be a minimal weak* cusco from a topological space A into subsets of the dual of a Banach space X, and let K be a non-empty weak* closed and convex subset of X^*. If for each non-empty open subset U of A, $\Phi(U) \not\subseteq K$ then $\{t \in S : \Phi(t) \cap K = \emptyset\}$ is a dense open subset of A.*

Proof. Let $W \equiv \{t \in A : \Phi(t) \cap K = \emptyset\}$. Since Φ is weak* upper semicontinuous and K is weak* closed, W is open. So it is sufficient to show that W is dense in A. To this end, let V be a non-empty open subset of A. Then $\Phi(V) \not\subseteq K$ so from Proposition 1.3 there exists a non empty open subset V' of V such that $\Phi(V') \cap K = \emptyset$ and so $\emptyset \neq V' \subseteq W \cap V$. Therefore W is dense in A.

THEOREM 1.5. *Let U be a non-empty open subset of a Baire space A and X a Banach space. Consider a minimal weak* cusco Φ from A into subsets of X^*. If for some countable family $\{B_n : n \in \mathbb{N}\}$ of weak* closed* *and convex subsets of* X^* *we have that* $\Phi(t) \cap \bigcup\{B_n : n \in \mathbb{N}\} \neq \emptyset$ *for each* t *in a second category subset* D *of* U, *then for some* $k \in \mathbb{N}$ *there exists a non-empty open subset* V *of* U *such that* $\Phi(V) \subseteq B_k$.

Proof. Let Φ' be the restriction of Φ to U. It follows from Proposition 1.3 that Φ' is a minimal weak* cusco on U. We note also, that U with the relative topology is a Baire space. Now if $\Phi'(W) \subseteq B_1$ for some non-empty open subset W of U write $V \equiv W$, but if not, we have by Proposition 1.4 that there is a dense open subset O_1 of U such that $\Phi'(O_1) \cap B_1 = \emptyset$. Now if $\Phi'(W) \subseteq B_2$ for some non-empty open subset W of U write $V \equiv W$, but if not, we have by Proposition 1.4 that there is a dense open subset O_2 of U such that $\Phi'(O_2) \cap B_2 = \emptyset$. Continue in this way. We will have defined V at some stage, because if not, we will have a dense G_δ subset O_∞ of U, where $O_\infty \equiv \bigcap\{O_n : n \in \mathbb{N}\}$ and $\Phi'(O_\infty) \cap \bigcup\{B_n : n \in \mathbb{N}\} = \emptyset$, contradicting the fact that for each $t \in O_\infty \cap D \neq \emptyset$ $\Phi(t) \cap \bigcup\{B_n : n \in \mathbb{N}\} \neq \emptyset$. So U contains a non-empty open subset V, such that for some $k \in \mathbb{N}$, $\Phi(V) \subseteq B_k$.

COROLLARY 1.6. *Let* A *be a Baire space,* X *a Banach space, and* Φ *a minimal weak* cusco from* A *into subsets of* X^*. *If for some separable subset* C *of* X^* *we have that* $\Phi(t) \cap C \neq \emptyset$ *for each* t *in a second category subset* D *of* A, *then for each* $\varepsilon > 0$ *there exists a non-empty open subset* V *of* A *such that* diam $\Phi(V) < \varepsilon$.

2. Some Geometrical Properties of Banach Spaces

In this section we shall apply Corollary 1.6 to obtain some new topological characterisations of some geometrical properties possessed by (separable) Banach spaces.

Let C be a non-empty bounded subset of a Banach space X. We say that a point $x \in C$ is a *point of continuity* of (C, weak) if the weak and norm topologies agree at x. Furthermore, for a non-empty bounded subset C of X^* we say that a point $f \in C$ is a *weak* point of continuity* of (C, weak*) if the weak* and norm topologies agree at f.

PROPOSITION 2.1. *Let* C *be a (convex) weak* compact subset of the dual of a Banach space* X. *Then the weak* points of continuity of* (C, *weak*) are residual in* (C, *weak*) if and only if, each non-empty (convex) relatively weak* open subset of* C *possesses non-empty relatively weak* open subsets of arbitrarily small diameter.*

Proof. Suppose that W is a non-empty (convex) relatively weak* open subset of C and that the weak* points of continuity of (C, weak*) are residual in (C, weak*). Then clearly W contains a weak* point of continuity of (C, weak^*), and so W possesses relatively weak* open subsets of arbitrarily small diameter.

Conversely, given $\varepsilon > 0$ consider the following open subset of C: $O_\varepsilon \equiv \bigcup\{\text{weak}^* \text{ open sets } U \text{ in } C : \operatorname{diam} U < \varepsilon\}$; we will show that O_ε is dense in (C, weak^*). To see this, let V be a non-empty relatively weak* open subset of C; note that if C is convex then without loss of generality we may assume that V is also convex. Now, by the hypothesis V contains a non-empty relatively weak* open subset U of V with diameter less than ε. Hence $\varnothing \neq U \subseteq O_\varepsilon \cap V$, and so O_ε is dense in C. The proof is completed by observing that each element of $\bigcap\{O_{1/n} : n \in \mathbb{N}\}$ is a weak* point of continuity of (C, weak^*).

PROPOSITION 2.2. *Let C be a non-empty closed bounded (convex) subset of a Banach space X. Then each non-empty relatively weak open (convex) subset of C possesses non-empty relatively weak open subsets of arbitrarily small diameter if, and only if, each non-empty relatively weak* open (convex) subset of $\overline{\hat{C}}^{w^*}$ possesses non-empty relatively weak* open subsets of arbitrarily small diameter.*

Proof. The proof of this result comes from the following three observations.

(1) For each relatively weak* open subset U of $\overline{\hat{C}}^{w^*}$, $U \cap \hat{C}$ is weak* dense in U;

(2) for each non-empty bounded subset U of X^{**} $\operatorname{diam} U = \operatorname{diam} \overline{U}^{w^*}$;

(3) each relatively weak open subset U of C extends to be a relatively weak* open subset $\tilde{U}$ of $\overline{\hat{C}}^{w^*}$ such that $\tilde{U} \cap \hat{C} = \hat{U}$.

We say that a Banach space X has the *convex point of continuity property* (or CPCP for short) if for each non-empty closed, bounded, convex subset C of X and each $\varepsilon > 0$ there exists a non-empty relatively weak open subset of C with diameter less than ε. Furthermore, we say that the dual of a Banach space X has the *weak* convex point of continuity property* (or C*PCP for short) if for each non-empty weak* compact, convex subset C of X^* and each $\varepsilon > 0$ there exists a non-empty relatively weak* open subset of C with diameter less than ε [8].

A notion very similar to that of the convex point of continuity property has also been considered [12]. We say that a Banach space X has the *point of continuity property* (or PCP for short) if for each non-empty bounded subset C of X and $\varepsilon > 0$ there exists a non-empty relatively weak open subset of C with diameter less than ε. Similarly, we say that the dual of a Banach space has the *weak* point of continuity property* (or P*CP for short) if for each non-empty bounded subset C of X^* and $\varepsilon > 0$ there exists a non-empty relatively weak* open subset of C with diameter less than ε.

THEOREM 2.3. *Let X be a separable Banach space. Then X has the point of continuity property (convex point of continuity property) if and only if, for each non-empty closed, bounded (convex) subset C of X, $\hat{C}$ is second category in $(\overline{\hat{C}}^{w^*}$, weak*).*

Proof. Suppose that X has the point of continuity property (convex point of continuity property) and that C is a closed and bounded (convex) subset of X. It follows from Proposition 2.2 that each non-empty (convex) relatively weak* open subset of $\overline{\hat{C}}^{w^*}$ possesses non-empty relatively weak* open subsets of arbitrarily small diameter. Therefore from Proposition 2.1 the set of weak* points of continuity of $(\overline{\hat{C}}^{w^*}$, weak*) are residual; and so second category in $(\overline{\hat{C}}^{w^*}$, weak*).

However each weak* point of continuity of $(\overline{\hat{C}}^{w^*}$, weak*) must lie in $\hat{C}$, and so we are done.

The converse statement follows immediately from Corollary 1.6.

THEOREM 2.4. *Let X be a separable Banach space. Then X^* has the weak* point of continuity property (weak* convex point of continuity property) if and only if, for each non-empty weak* compact (convex) subset C of X^* the points where the weak* and weak topologies agree are second category in (C, weak*).*

Proof. It follows directly from Proposition 2.1 that if X^* has the weak* point of continuity property (weak* convex point of continuity) then for every weak* compact (convex) subset C of X^* the points where the weak and weak* topologies agree are second category in (C, weak*); in fact they are residual in (C, weak*). Conversely, suppose that C is a non-empty weak* compact (convex) subset of X^* and that the set S of points where the weak and weak* topologies agree is second category in (C, weak*). Now, since X is separable (C, weak*) is metrisable, and hence separable, and furthermore (S, weak*) is also separable. Let $\{f_n : n \in \mathbb{N}\}$ be a dense subset of (S, weak*), and let $F \equiv \overline{\text{co}}\{f_n : n \in \mathbb{N}\}$. Clearly S is contained in F, because if there exists an element $t \in S \backslash F$ then there exists a weak* open neighbourhood U of t such that $U \cap F \neq \varnothing$ (since the weak and weak* topologies agree at t). However, this contradicts the fact that the set $\{f_n : n \in \mathbb{N}\}$ is dense in (S, weak*). The conclusion now follows from Corollary 1.6).

We end this section with the following example.

EXAMPLE 2.5. Consider $X \equiv c_0(\mathbb{N})$ with its usual norm. Then $B(\hat{X})$ is first category in ($B(X^{**})$, weak*).

Proof. It is easy to check that there are no non-empty relatively weak open subsets of $(B(X)$, weak) with diameter less than 2. Therefore Corollary 1.6 tells us that $B(\hat{X})$ cannot be second category in $(B(X^{**})$, weak*).

On the other hand, it is well known that $c_0(\mathbb{N})$ can be equivalently renormed to be locally uniformly rotund, and in this case, Proposition 2.1 and Proposition 2.2. together tell us that the natural embedding of this equivalent norm ball into its second dual ball is residual with respect to the relative weak* topology.

3. On Weak and Weak* Fragmentability of Banach Spaces

In this section we extend the results given in Section two to non-separable Banach spaces. In doing so we obtain an interesting variation on Goldstine's theorem.

THEOREM 3.1. *For a non-empty closed, bounded (convex) subset C of a Banach space X the following properties are equivalent.*

(1) *The points in $\overline{\hat{C}}^{w^*}$ where the weak and weak* topologies of $\overline{\hat{C}}^{w^*}$ agree are residual in $(\overline{\hat{C}}^{w^*}$, weak*).*

(2) *$\hat{C}$ is residual in $(\overline{\hat{C}}^{w^*}$, weak*).*

(3) *The weak* points of continuity of $(\overline{\hat{C}}^{w^*}$, weak*) are residual in $(\overline{\hat{C}}^{w^*}$, weak*).*

(4) *The points of continuity of $(C$, weak) contain a dense G_δ subset of $(C$, weak).*

(5) *Each non-empty (convex) relatively weak* open subset of $\overline{\hat{C}}^{w^*}$ possesses non-empty relatively weak* open subsets of arbitrarily small diameter.*

(6) *Each non-empty (convex) relatively weak open subset of C possesses non-empty relatively weak open subsets of arbitrarily small diameter.*

Proof. That (1) implies (2) comes from the observation that each point, where the weak and weak* topologies agree, lies in $\hat{C}$.

That (2) implies (3) follows immediately from Theorem 1.2.

That (3) is equivalent to (4) comes from noticing that a point $F \in \overline{\hat{C}}^{w^*}$ is a weak* point of continuity of $(\overline{\hat{C}}^{w^*}$, weak*) if and only if, F is a member of $\hat{C}$ and F is a point of continuity of $(\hat{C}$, weak).

That (4) is equivalent to (5) comes from Proposition 2.1.

Finally, from Proposition 2.2 we see that (5) is equivalent to (6), and so the proof is completed by noticing that (3) implies (1).

COROLLARY 3.2. *A Banach space X has the point of continuity property (convex point of continuity property) if and only if for each non-*

empty closed bounded (convex) subset C of X, $\hat{C}$ is residual in $(\overline{\hat{C}}^{w^*}$, weak).*

The following corollary answers a natural question raised by Goldstine's theorem. Namely, when is the natural embedding of $B(X)$ into $B(X^{**})$ not only dense, but also residual in $(B(X^{**})$, weak*)?

COROLLARY 3.3 *Let $B(X)$ be an equivalent norm ball on a Banach space X. Then $B(\hat{X})$ is residual in $(B(X^{**})$, weak*) if and only if, the points of continuity of $(B(X)$, weak) contain a dense G_δ subset of $(B(X)$, weak).*

PROPOSITION 3.4. *A Banach space X has the convex point of continuity property, if and only if, each equivalent norm ball on X possesses relatively weak open subsets of arbitrarily small diameter.*

Proof. Clearly if X has the convex point of continuity property then each equivalent norm ball possesses non-empty relatively weak open subsets of arbitrarily small diameter. Now we consider the converse.

Suppose that $\varepsilon > 0$ is given and that C is a non-empty closed bounded convex subset of X. It will be sufficient to show that C possesses a non-empty relatively weak open subset of diameter less than ε. If $B(X)$ is an equivalent norm ball on X, then so is $B_1(X) \equiv \overline{B(X) + (-C) + C}$. Now let W be a weak open subset of X such that $W \cap B_1(X) \neq \emptyset$ and $\operatorname{diam}(W \cap B_1(X)) < \varepsilon$. Clearly for some $x \in B(X)$ and $y \in -C$, $(\{x + y\} + C) \cap W \neq \emptyset$ and so $C \cap (W - \{x + y\}) \neq \emptyset$; but clearly $\operatorname{diam}(C \cap (W - \{x + y\})) < \varepsilon$ and so we are done.

THEOREM 3.5. *A Banach space X has the convex point of continuity property if and only if, for each equivalent norm ball on X, $B(\hat{X})$ is residual in $(B(X^{**})$, weak*).*

Proof. It is clear from Theorem 3.1 that if X has the convex point of continuity property then for each equivalent norm ball $B(X)$ on X, $B(\hat{X})$ will be residual in $(B(X^{**})$, weak*). On the other hand, if for each equivalent norm ball $B(X)$ on X $B(\hat{X})$ is residual in $(B(X^{**})$, weak*) then by Theorem 3.1 $B(X)$ will possess non-empty relatively weak open subsets of arbitrarily small diameter, and so from Proposition 3.4 X will have the convex point of continuity property.

We now consider a dual version of this theorem.

THEOREM 3.6. *The dual of a Banach space X has the weak* point of continuity property (weak* convex point of continuity property) if and only if, for each non-empty weak* compact (convex) subset C of X^* the points where the weak and weak* topologies agree are residual in $(C$, weak*).*

We note that by repeating the argument used in Proposition 3.4, we could, in the theorem above, prove that X^* has the weak* convex point of continuity property if and only if, for each equivalent dual ball $B(X^*)$ the points where the weak and weak* topologies agree are residual in $(B(X^*)$, weak*).

4. On a Characterisation of the Radon–Nikodym Property

Consider a non-empty bounded subset E of a Banach space X.

Given $f \in X^*\backslash\{0\}$ and a $\delta > 0$ the *slice* of E defined by f and δ is the subset $S(E, f, \delta) \equiv \{x \in E : f(x) > \sup(E) - \delta\}$. When E lies in the dual of a Banach space, and the slicing functional f given above is weak* continuous, then we call the slice of E a *weak* slice* of E.

We say that a Banach space X has the *Radon–Nikodym Property* (or RNP for short) if each non-empty bounded subset E of X possesses a slice of arbitrarily small diameter.

In order to prove our main theorem we will need a few basic results concerning the differentiability of the norm on a Banach space.

Let X be a Banach space and let $\|\cdot\|$ be an equivalent dual norm on X^*. The *subdifferential* of $\|\cdot\|$ at $f \in X^*$ is denoted by $\partial\|f\|$ and is the set $\{F \in S(X^{**}) : F(f) = \|f\|\}$. The *subdifferential mapping* $f \rightarrow \partial\|f\|$ is a minimal weak* cusco from X^* into subsets of $S(X^{**})$ [13, p. 100]. The next proposition establishes a close connection between the subdifferential mapping and the geometry of the corresponding second dual ball. For a proof of this result see [9].

PROPOSITION 4.1. *Let X be Banach space, and let $\|\cdot\|$ be an equivalent dual norm on X^*. Then for each $f \in X^*$ and $\delta > 0$, $S(B(X^{**}), \hat{f}, \delta^2) \subseteq \partial\|B(f, \delta)\| + \delta B(X^*)$.*

The next theorem is a slight variation on a well-known characterisation of spaces which have the Radon–Nikodym property [4].

THEOREM 4.2. *A Banach space X has the Radon–Nikodym property if and only if, for each equivalent dual norm $\|\cdot\|$ on X^* the subdifferential mapping $f \rightarrow \partial\|f\|$ is single-valued and norm upper semi-continuous on a dense G_δ subset of X^*.*

Proof. The proof of this theorem comes straight from Corollary 3 and Theorem 4 in [4] and the result that the dual norm $\|\cdot\|$ on X^* is Frechet differentiable at $g \in X^*$ if and only if, the subdifferential mapping $f \rightarrow \partial\|f\|$ is single-valued and norm upper semi-continuous at g [13, p. 100].

It is a straightforward consequence of the Bishop–Phelps theorem [1] that wherever the subdifferential mapping $f \rightarrow \partial\|f\|$ is single-valued and norm upper semi-continuous its subdifferential lies in $\hat{X}$. So from Theorem 4.2 we get the following corollary.

COROLLARY 4.3. *If a Banach space X has the Radon–Nikodym property then for each equivalent dual norm $\|\cdot\|$ on X^* the set $\{f \in X^* : \partial\|f\| \cap \hat{X} \neq \varnothing\}$ is residual in X^*.*

THEOREM 4.4. *A Banach space X has the Radon–Nikodym property if and only if, for each equivalent dual norm $\|\cdot\|$ on X^* the set $\{f \in X^* : \partial\|f\| \cap \hat{X} \neq \varnothing\}$ is residual in X^*.*

Proof. It is immediate from Corollary 4.3 that if X has the Radon–Nikodym property then for each equivalent norm $\|\cdot\|$ on X the corresponding set $\{f \in X^* : \partial\|f\| \cap \hat{X} \neq \varnothing\}$ is residual in X^*. We now consider the converse. It is well known that a Banach space X has the Radon–Nikodym property if each non-empty closed, bounded, convex, and separable subset of X possesses slices of arbitrarily small diameter [3, p. 31]. This is the approach that we shall adopt. So let C be a non-empty closed, bounded, convex, and separable subset of X, and suppose $\varepsilon > 0$ is given. After possibly re-scaling, we may assume that $\sup\{\|x\| : x \in C\} = 2$. Let $K \equiv \overline{\mathrm{co}}\{C \cup -C\}$ and let $B(X)$ be an equivalent norm ball on X. It is not too difficult to see that if K possesses a slice with a diameter less than ε, then so does C. So our goal from here will be to construct a slice of K with a diameter less than ε. We being by noticing that $B_1(X) \equiv \overline{\mathrm{co}}\{B(X) \cup K\}$ is an equivalent norm ball on X and that $B_1(X^{**}) \equiv \mathrm{co}\{B(X^{**}) \cup \hat{K}^{w^*}\}$. Now choose $z \in B_1(X)\backslash B(X)$ and let $f \in X^*$ strongly separate z from $B(X)$. We can now select a $\delta > 0$ such that $z \in S(B_1(X), f, \delta)$ and $S(B_1(X), f, \delta) \cap B(X) = \varnothing$. Clearly $\partial\|f\| \subseteq S(B_1(X^{**}), \hat{f}, \delta)$ and so by the weak* upper semi-continuity of $\partial\|\cdot\|$ there exists an open neighbourhood U of f such that $\partial\|U\| \subseteq S(B_1(X^{**}), \hat{f}, \delta)$. However, we must in actual fact have that $\partial\|U\| \subseteq \hat{K}^{w^*}$, since the images of $\partial\|\cdot\|$ are extremal subsets of $B_1(X^{**})$. We now observe that $G \cap U$ is residual in U, and that for each $g \in G \cap U$ $\partial\|g\| \cap \hat{K} \neq \varnothing$. So by Corollary 1.6 there exists a non-empty open subset V of U such that $\mathrm{diam}(\partial\|V\|) < \varepsilon/3$. Choose $h \in V\backslash\{0\}$ and $0 < r < \varepsilon/3$ such that $B(h, r) \subseteq V$. Hence by Proposition 4.1 $\mathrm{diam}\, S(B_1(X^{**}), \hat{h}, r^2) < \varepsilon/3 + 2(\varepsilon/3) = \varepsilon$, from which it follows that $\mathrm{diam}\, S(K, h, r^2) < \varepsilon$, and we are done.

Let $\|\cdot\|$ be an equivalent norm on a Banach space X. Then we call the set $\{f \in S(X^*) : f \text{ attains its norm on } B(X)\}$ the *Bishop–Phelps* set. From the Bishop–Phelps theorem [1] we know that this set is always dense in $S(X^*)$.

Theorem 4.4 may now be written in terms of the Bishop–Phelps set.

THEOREM 4.5. *A Banach space X has the Radon–Nikodym property if* *and only if, for each equivalent norm on X the corresponding Bishop–Phelps set is residual in $S(X^*)$.*

Proof. This result follows from the fact that for an equivalent norm ball $B(X)$ on X, $f \in S(X^*)$ attains its norm on $B(X)$ if and only if, $\partial\|f\| \cap \hat{X} \neq \varnothing$.

5. More on the Point of Continuity Property

In this section we extend the characterisation given in Theorem 3.1, by exploiting some recent continuity properties considered in [7, 10].

Let f be a function from a topological space A into a Banach space X. We say that f is *ω continuous* at $t \in A$ if for each $\varepsilon > 0$ there exists a weak compact set K in X and an open neighbourhood U of t such that $f(U) \subseteq K + \varepsilon B(X)$. Extending this notion further, we say that f is *γ continuous* at $t \in A$ if for each $\varepsilon > 0$ there exists a countable family $\{K_n : n \in \mathbb{N}\}$ of weak compact subsets of X and an open neighbourhood U of t such that $f(U) \subseteq \bigcup\{K_n : n \in \mathbb{N}\} + \varepsilon B(X)$. The significance of these continuity properties is revealed in the next proposition.

Propoposition 5.1. *For a weak* continuous function f, from a Baire space A into the dual of a Banach space X, the following are equivalent.*

(1) *f is norm continuous on a residual subset of A.*
(2) *f is ω continuous on a residual subset of A.*
(3) *f is γ continuous on a residual subset of A.*

Proof. It is clear that (1) implies (2) and (2) implies (3), so it is sufficient to show that (3) implies (1). However, (3) implies (1) comes from Theorem 4 in [10].

Let C be a non-empty bounded subset of a Banach space X. We say that $x \in C$ is a *point of ω continuity* of (C, weak) if for each $\varepsilon > 0$ there exists a weak compact set K in X and a weak open set W of X such that $x \in C \cap W \subseteq K + \varepsilon B(X)$. Similarly, we say that a point $x \in C$ is a *point of γ continuity* of (C, weak) if for each $\varepsilon > 0$ there exists a countable family $\{K_n : n \in \mathbb{N}\}$ of weak compact subsets of X and a weak open set W of X such that $x \in C \cap W \subseteq \bigcup\{K_n : n \in N\} + \varepsilon B(X)$. Furthermore, for a non-empty bounded subset C of X^* we say that a point $f \in C$ is a *weak* point of ω continuity* of (C, weak*) if for each $\varepsilon > 0$ there exists a weak compact subset K of X^* and a weak* open set W of X^* such that $f \in C \cap W \subseteq K + \varepsilon B(X^*)$. Finally, we say that a point $f \in C$ is a weak* point of γ continuity of (C, weak*) if for each $\varepsilon > 0$ there exists a countable family $\{K_n : n \in \mathbb{N}\}$ of weak compact subsets of X^* and a weak* open set W of X^* such that $f \in C \cap W \subseteq \bigcup\{K_n : n \in \mathbb{N}\} + \varepsilon B(X^*)$.

PROPOSITION 5.2. *Let C be a non-empty closed, bounded subset of a Banach space X. If a point $x \in C$ is a point of ω continuity of $(C$, weak) then $\hat{x}$ is a weak* point of ω continuity of $\overline{\hat{C}}^{w^*}$, weak*).*

Proof. Suppose that $x \in C$ is a point of ω continuity of $(C$, weak) and that $\varepsilon > 0$ is given. We can choose a relatively weak* open subset W of $\overline{\hat{C}}^{w^*}$ such that $\hat{x} \in \hat{C} \cap W \subseteq \hat{K} + \varepsilon B(\hat{X})$ for some weak compact subset K of X. Now $W \cap \hat{C}$ is weak* dense in W, so $x \in W \subseteq \overline{W \cap \hat{C}}^{w^*} \subseteq \hat{K} + \varepsilon B(X^{**})$, since $\hat{K} + \varepsilon B(X^{**})$ is weak* closed. Therefore $\hat{x}$ is a weak* point of ω continuity of $(\overline{\hat{C}}^{w^*}$, weak*).

THEOREM 5.3. *For a non-empty closed, bounded subset C of a Banach space X the following properties are equivalent.*

(1) *The points of continuity of $(C$, weak) contain a dense G_δ subset of $(C$, weak).*

(2) *The points of ω continuity of $(C$, weak) contain a dense G_δ subset of $(C$, weak).*

(3) *The weak* points of ω continuity of $(\overline{\hat{C}}^{w^*}$, weak*) are residual in $(\overline{\hat{C}}^{w^*}$, weak*).*

(4) *The weak* points of γ continuity of $(\overline{\hat{C}}^{w^*}$, weak*) are residual in $(\overline{\hat{C}}^{w^*}$, weak*).*

(5) *The weak* points of continuity of $(\overline{\hat{C}}^{w^*}$, weak*) are residual in $(\overline{\hat{C}}^{w^*}$, weak*).*

Proof. It is obvious that (1) implies (2). That (2) implies (3) comes from Proposition 5.2. That (4) implies (5) comes from Proposition 5.1. Finally, from Theorem 3.1, we see that (5) implies (1).

Remark 5.4. We note that for a function f from a topological space A into a Banach space X, the points where f is continuous, with respect to the norm, ω or γ always form a G_δ subset of A. Hence in Theorem 3.1 and Theorem 5.3, we may replace "dense G_δ" by simply dense.

We now show that, even if every point of $B(X)$ is a point of γ continuity, $B(X)$ may contain no points of continuity.

EXAMPLE 5.5. Consider $X \equiv c_0(\mathbb{N})$ with its usual norm. Then each point of $B(X)$ is a point of γ continuity of $(B(X)$, weak). However, no point of $B(X)$ is a point of continuity of $(B(X)$, weak).

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