Closed graph theorems and Baire spaces

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Abstract. In this note we consider the question of when a nearly continuous function acting between topological spaces is continuous. In doing so we obtain a topological version of the classical closed graph theorem and a topological version of the Banach-Steinhaus theorem.

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1 Introduction

Let $f : X \to Y$ be a function acting between topological spaces X and Y. We say that f is *nearly continuous* on X if for each open set U in Y, $f^{-1}(U) \subseteq \text{int}\overline{f^{-1}(U)}$. Clearly every continuous mapping is nearly continuous. In this note we consider the converse question, *ie*: when is a given nearly continuous function continuous?

We begin in Section 2 by summarising some of the known results in terms of the notion of "very near continuity". Then in Section 3 we use the notion of "very near continuity" to show that every nearly continuous mapping with closed graph (or nearly G_δ graph) acting from a Baire space into a partition complete (strong Choquet) space is continuous.

2 Very nearly continuous mappings

Let $f : X \to Y$ be a mapping acting between topological spaces X and Y. We will say that f has *property P with respect to X and Y* if for each pair of disjoint open sets U and V in Y, $\text{int}\overline{f^{-1}(U)} \cap \text{int}\overline{f^{-1}(V)} = \emptyset$. Clearly if f is continuous then f has property P. However, if f is nearly continuous and satisfies property P then for each open set U in Y, $f^{-1}(U) \subseteq \text{int}\overline{f^{-1}(U)} \subseteq f^{-1}(\overline{U})$. Therefore we have the following result.

Proposition 1 *Let $f : X \to Y$ be a nearly continuous mapping acting from a topological space X into a regular topological space Y. Then f is continuous on X if, and only if, f satisfies property P with respect to X and Y.*

Henceforth we shall be concerned with conditions on X, Y and f that imply property P. Let $f : X \to Y$ be a mapping acting between topological spaces X and Y. We shall say that f is *very nearly continuous* on X if for each open set U in Y, $f^{-1}(U)$ is a residual subset of $\text{int}\overline{f^{-1}(U)}$.

Theorem 1 *Every very nearly continuous mapping acting between a Baire space X and a topological space Y satisfies property P with respect to X and Y.*

Proof: Suppose U and V are open subsets of Y. If $W := \text{int}\overline{f^{-1}(U)} \cap \text{int}\overline{f^{-1}(V)} \neq \emptyset$ then $f^{-1}(U) \cap W$ and $f^{-1}(V) \cap W$ are residual subsets of W and so,

$$\emptyset \neq [f^{-1}(U) \cap W] \cap [f^{-1}(V) \cap W] = [f^{-1}(U) \cap f^{-1}(V)] \cap W = f^{-1}(U \cap V) \cap W \subseteq f^{-1}(U \cap V).$$

Therefore, $U \cap V \neq \emptyset$; which shows that f satisfies property P. □

A mapping $f : X \to Y$ acting between topological spaces X and Y is said to be *G_δ-continuous (Borel measurable of class one)* if $f^{-1}(U)$ is a G_δ subset of X for each open (closed) subset of X.

Corollary 1 ([11], Theorem 3) *Let $f : X \to Y$ be a nearly continuous mapping acting between a Baire space X and a regular space Y. If f is either G_δ-continuous or Borel measurable of class one then f is continuous.*

Proof: If f is G_δ-continuous then for each open set U in Y, $f^{-1}(U)$ is a dense G_δ subset of $\text{int}\overline{f^{-1}(U)}$ and so f is very nearly continuous. If f is Borel measurable of class one then for each open set V in Y, $f^{-1}(\overline{V}) \cap \text{int}\overline{f^{-1}(V)}$ is a dense G_δ subset of $\text{int}\overline{f^{-1}(V)}$. Therefore, for any open set U in Y, $f^{-1}(U)$ is locally residual in $\text{int}\overline{f^{-1}(U)}$ and so residual in $\text{int}\overline{f^{-1}(U)}$. This shows that f is very nearly continuous. □

A mapping $f : X \to Y$ acting between topological spaces X and Y that is the pointwise limit of a sequence $(f_n : n \in \mathbb{N})$ of continuous mappings is said to be of *Baire class one*. Now if U is any open subset of Y then,

$$f^{-1}(U) \subseteq \bigcap_{n\in\mathbb{N}} \Big(\bigcup_{k\geq n} f_k^{-1}(U) \Big) \subseteq f^{-1}(\overline{U})$$

and so $f^{-1}(\overline{U}) \cap \text{int}\overline{f^{-1}(U)}$ is a residual subset of $\text{int}\overline{f^{-1}(U)}$.

Corollary 2 ([11], Theorem 4) *Every nearly continuous Baire class one mapping acting from a Baire space into a regular space is continuous.*

Proof: It follows from the above observation that for each open set U in Y, $f^{-1}(U)$ is locally residual in $\text{int}\overline{f^{-1}(U)}$ and so residual in $\text{int}\overline{f^{-1}(U)}$. □

Remark: The previous two corollaries simplify and make more transparent the proofs of Theorems 3 and 4 in [11].

A function $f : X \to Y$ acting between topological spaces X and Y is said to be *almost continuous* on X if for each open set U in Y, $f^{-1}(U)$ is an everywhere second category subset of $\text{int}\overline{f^{-1}(U)}$. Clearly every almost continuous mapping is nearly continuous and if X is a Baire space then every very nearly continuous mapping is almost continuous. So almost continuity lies between near continuity and very near continuity. If in addition f has the property that the inverse image of each open set in Y has the Baire property in X then almost continuity implies very near continuity. Hence we obtain the following result from [17].

Corollary 3 *Every almost continuous mapping from a Baire space into a regular space which is lower Baire (ie: inverse images of open sets are sets with the Baire property) is continuous.*

3 Closed graph theorem

In order to prove our closed graph theorem we need to introduce some more definitions. A sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of covers of a topological space X is said to be *complete* if each filter base $\mathcal{F}$ on X that is $\mathcal{V}_n$-small for each $n \in \mathbb{N}$ (*ie:* for each $n \in \mathbb{N}$ there exists a $V_n \in \mathcal{V}_n$ containing some $F \in \mathcal{F}$) has $\bigcap_{F\in\mathcal{F}} \overline{F} \neq \emptyset$. A cover $\mathcal{V}$ of X is call *exhaustive* provided every nonempty subset A of X has a nonempty relatively open subset of the form $A \cap V$ with $V \in \mathcal{V}$ and a regular space X is called *partition complete* or *cover complete*, [10] if it has a complete sequence of exhaustive covers. In addition to these definitions we also need to consider the Banach-Mazur game. Let X be a topological space and let R be a subset of X. On X we consider the *BM(R)-game* played between two players α and β. A *play* of the $BM(R)$-game is a decreasing sequence of non-empty open sets $A_n \subseteq B_n \subseteq \cdots B_2 \subseteq A_1 \subseteq B_1$ which have been chosen alternatively; the A_n's by α and the B_n's by β. The player α is said to have *won* a play of the $BM(R)$-game if $\bigcap_{n\in\mathbb{N}} A_n \subseteq R$; otherwise β is said to have won. A *strategy* s for the player α is a "rule" that tells him/her how to play (possibly depending on all the previous moves of β). Since the move of α may depend on the previous moves of β we shall denote the n-th move of α by, $s(B_1, B_2, \ldots B_n)$. We say that s is a *winning strategy* if, using it, he/she wins every play, independently of player β's choices. (A more detailed description of the Banach-Mazur game and a proof of the next lemma may be found in [13].)

Lemma 1 *Let R be a subset of a topological space X. Then R is residual in X if, and only if, the player α has a winning strategy in the $BM(R)$-game played on X.*

Theorem 2 *Every nearly continuous mapping with closed graph acting from a Baire space into a partition complete space is continuous.*

Proof: Let $f : X \to Y$ be a nearly continuous mapping with closed graph acting from a Baire space X into a partition complete space Y (with associated complete sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of exhaustive covers). It will, in light of Theorem 1, suffice to show that f is very nearly continuous. To this end, let U be any non-empty open subset of Y. We shall construct a winning strategy for the player α in the $BM(f^{-1}(U))$-game played in $\mathrm{int}\overline{f^{-1}(U)}$. Suppose that β's first move is the non-empty open subset B_1 of $\mathrm{int}\overline{f^{-1}(U)}$. Player α's response to this is the following. First α chooses an open set U_1 such that $\emptyset \neq D_1 := f(B_1)\cap U_1 \subseteq U_1 \subseteq \overline{U_1} \subseteq U$ and $D_1 \subseteq V_1$ for some $V_1 \in \mathcal{V}_1$. Then he/she defines $A_1 := \mathrm{int}\overline{f^{-1}(U_1)} \cap B_1 \neq \emptyset$ and notes that $A_1 \subseteq \overline{f^{-1}(D_1)}$. In general, if β selects a non-empty open subset $B_{n+1} \subseteq A_n \subseteq \cdots A_1 \subseteq B_1$ then the player α responds in the following way. First he/she chooses an open set $U_{n+1} \subseteq U_n$ such that $\emptyset \neq D_{n+1} := f(B_{n+1}) \cap U_{n+1} \subseteq f(B_n) \cap U_n = D_n$ and $D_{n+1} \subseteq V_{n+1}$ for some $V_{n+1} \in \mathcal{V}_{n+1}$. Then he/she defines $A_{n+1} := \mathrm{int}\overline{f^{-1}(U_{n+1})} \cap B_{n+1} \neq \emptyset$ and notes that $A_{n+1} \subseteq \overline{f^{-1}(D_{n+1})}$. With this strategy we see that:

$$\bigcap_{n\in\mathbb{N}} A_n \subseteq \bigcap_{n\in\mathbb{N}} \overline{f^{-1}(D_n)} \subseteq f^{-1}(\bigcap_{n\in\mathbb{N}} \overline{D_n}) \subseteq f^{-1}(U).$$

Note: the set inclusion $\bigcap_{n\in\mathbb{N}} \overline{f^{-1}(D_n)} \subseteq f^{-1}(\bigcap_{n\in\mathbb{N}} \overline{D_n})$ follows from the following argument. Let $x \in \bigcap_{n\in\mathbb{N}} \overline{f^{-1}(D_n)}$ and let $\mathcal{N}$ be any neighbourhood base for x. Then,
$\mathcal{F} := \{f(N) \cap D_n : N \in \mathcal{N} \text{ and } n \in \mathbb{N}\}$ is a filter base on Y that is $\mathcal{V}_n$-small for each $n \in \mathbb{N}$. Hence,

$$\begin{aligned}\emptyset \neq \bigcap\{\overline{F} : F \in \mathcal{F}\} &\subseteq \bigcap\{\overline{f(N)} : N \in \mathcal{N}\} \cap \bigcap\{\overline{D_n} : n \in \mathbb{N}\} \\ &= \{f(x)\} \cap \bigcap\{\overline{D_n} : n \in \mathbb{N}\} \quad \text{since } f \text{ has closed graph}\end{aligned}$$

ie: $f(x) \in \bigcap_{n\in\mathbb{N}} \overline{D_n}$ or equivalently, $x \in f^{-1}(\bigcap_{n\in\mathbb{N}} \overline{D_n})$. Therefore, $f^{-1}(U)$ is a residual subset of $\mathrm{int}\overline{f^{-1}(U)}$; which shows that f is very nearly continuous. □

The previous theorem improves Theorem 2 of [11] where the condition of "monotonely Čech-complete" (or equivalently, "strongly complete", ([11], p.142) or even "sieve-complete", ([10], p.114)) was used in place of the weaker hypothesis of partition complete (used here), [10]. The relationship between monotone Čech-completeness and partition completeness is discussed in [15]. Let us also take the opportunity to present some equivalent formulations of partition completeness. Let X be a topological space. On X we consider the $G(X)$-game played between two players I and II. Players I and II alternatively choose non-empty subsets $S_1 \supseteq T_1 \supseteq S_2 \supseteq T_2 \cdots$ of X such that T_n (chosen by II) is relatively open in S_n (chosen by I). Player II *wins* if $(T_n : n \in \mathbb{N})$ is a complete sequence of subsets of X, [10]. A strategy for the player II is defined in a similar way to a strategy for the player α in the $BM(R)$-game.

Proposition 2 *For a topological space (X, τ) the following are equivalent:*
(a) X is partition complete;
(b) X has a complete exhaustive sieve (see, [10] for definition);
(c) Player II has a winning strategy in the $G(X)$-game played on X;
(d) there exists a pseudo-metric d on X that "fragments" X and has the property that every d-Cauchy filter base on X has a τ-cluster point in X (see, [8] for definition of fragment).

Proof: The fact that (a) and (b) are equivalent follow from Propoistion 2.1 in [10] and the equivalence of (a) and (c) is Theorem 7.3 in [9]. Furthermore it is easy to see that (d) implies (c) so it remains to justify that (c) implies (d). However, this is very similar to Theorem 1.2 in [8]. □

Next we give a concrete example of a partition complete space that is not monotonely Čech-complete.

Example: Let $X := \ell^1(\mathbb{N})$ and let B_X be the unit ball in X. Then it is well known that the natural metric on $\ell^1(\mathbb{N})$ fragments (B_X, weak), ([5], p.7) but that $0 \in B_X$ is not a q-point of (B_X, weak). Since if $0 \in B_X$ were a q-point then one could show that 0 has a countable local base in (B_X, weak) and so conclude that $X^* = \ell^\infty(\mathbb{N})$ is separable ([4], V. 5.2); which it is not. Hence, (B_X, weak) is partition complete but not monotonely Čech-complete. □

It is now natural to ask if there are any other topological conditions on the graph of a nearly continuous function that imply continuity. The answer is "yes" and our response to this question is patterned on Theorem 1 of [11]. Given a subset A of a topological space X we say that A is a *nearly* G_δ subset of X if there exists a sequence $\{O_n : n \in \mathbb{N}\}$ of open subsets of X such that $\bigcap_{n\in\mathbb{N}} O_n \subseteq A$ and each $O_n \cap A$ is dense in A. To formulate the statement of our theorem we need to consider another topological game. Let X be a topological space. The *strong Choquet game* on X ([7], p. 196) is played by two players α and β. Player β starts by choosing a non-empty open set B_1 and an element $b_1 \in B_1$. Player α then plays a non-empty open set A_1 with $b_1 \in A_1 \subseteq B_1$. β follows by choosing a non-empty open set B_2 and an element b_2 such that $b_2 \in B_2 \subseteq A_1$, etc. We say that α *wins* a play of the game if $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$; otherwise β wins. A *strategy* s for the player α is a "rule" that tells him/her how to play. Since the move of α may depend on the previous moves of β we shall denote the nth move of α by, $s((b_1, B_1), \ldots (b_n, B_n))$. We say that s is a *winning strategy* if α wins every play of the strong Choquet game where he/she follows the strategy s. A topological space X is call a *strong Choquet space* if α has a winning strategy in the strong Choquet game played on X. It is immediately clear from this definition that all locally Čech-complete spaces are strong Choquet spaces. However, there are many other interesting examples of strong Choquet spaces. Recall that a topological space X is called *pseudo-compact* if every real-valued function defined on

X is bounded. In particular, this means that every real-valued function defined on X attains its maximum value at some point $x_0 \in X$, because if $f : X \to \mathbb{R}$ does not attain its maximum value on X then $g : X \to \mathbb{R}$ defined by, $g(x) := 1/[\sup\{f(t) : t \in X\} - f(x)]$ is a continuous function on X that is not bounded above.

Remark: The strong Choquet game seems to date back to G. Choquet himself when he considered this game to prove the following theorem "Let X be a non-empty topological space, then X is Polish if, and only if, X is second countable, T_1, regular and strong Choquet.", ([7], p.197).

Example: Every completely regular pseudo-compact topological space X is a strong Choquet space. To show that X is a strong Choquet space we need to exhibit a winning strategy s for the player α in the strong Choquet game played on X. If β selects $b_1 \in B_1$ then α responds by choosing an open set A_1 and a function $f_1 : X \to [0,1]$ such that $b_1 \in A_1 \subseteq f_1^{-1}(1) \subseteq B_1$. Then α defines, $s((b_1, B_1)) := A_1$. In general, if β selects $b_n \in B_n \subseteq A_{n-1}$ then α responds by choosing an open set A_n and a function $f_n : X \to [0,1]$ such that $b_n \in A_n \subseteq f_n^{-1}(1) \subseteq B_n$. Then α defines, $s((b_1, B_1), \ldots (b_n, B_n)) := A_n$. Now, since X is pseudo-compact there exists a point $x_0 \in X$ such that the continuous function $g(x) := \sum_{n=1}^{\infty}(1/2^n) f_n(x)$ attains its maximum value. Then, $x_0 \in \bigcap_{n\in\mathbb{N}} f_n^{-1}(1) = \bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$ and so X is a strong Choquet space.

The following theorem improves Theorem 1 in [11].

Theorem 3 *Every nearly continuous mapping with a nearly G_δ graph acting from a Baire space into a strong Choquet space is continuous.*

Proof: Let $f : X \to Y$ be a nearly continuous mapping with a nearly G_δ graph (*ie*: there exists a sequence $\{O_n : n \in \mathbb{N}\}$ of open subsets of $X \times Y$ such that $\bigcap_{n\in\mathbb{N}} O_n \subseteq G(f)$ and each $O_n \cap G(f)$ is dense in $G(f)$) acting from a Baire space X into a strong Choquet space Y (with winning strategy s). As in Theorem 2 it will suffice to show that f is very nearly continuous. To this end, let U be any non-empty open subset of Y. We shall construct a winning strategy for the player α in the $BM(f^{-1}(U))$-game played in $\mathrm{int}\overline{f^{-1}(U)}$. Suppose that β's first move is the non-empty open subset B_1 of $\mathrm{int}\overline{f^{-1}(U)}$. Player α's response to this is the following. First α chooses x_1, V_1 and W_1 such that $(x_1, f(x_1)) \in V_1 \times W_1 \subseteq O_1 \cap [B_1 \times U]$. (Note: this is possible since $[B_1 \times U] \cap G(f) \neq \emptyset$.) Then he/she defines $U_1 := s((f(x_1), W_1))$ and $A_1 := V_1 \cap \mathrm{int}\overline{f^{-1}(U_1)} \neq \emptyset$ and notes that $A_1 \times U_1 \subseteq O_1$ and $A_1 \subseteq \mathrm{int}\overline{f^{-1}(U_1)}$. In general, if β selects a non-empty open subset $B_{n+1} \subseteq A_n \subseteq \cdots A_1 \subseteq B_1$ then the player α responds in the following way. First, he/she chooses x_{n+1}, V_{n+1} and W_{n+1} such that $(x_{n+1}, f(x_{n+1})) \in V_{n+1} \times W_{n+1} \subseteq O_{n+1} \cap [B_{n+1} \times U_n]$. (Note: this is possible since $[B_{n+1} \times U_n] \cap G(f) \neq \emptyset$.) Then he/she defines $U_{n+1} := s((f(x_1), W_1), \ldots (f(x_{n+1}), W_{n+1}))$ and $A_{n+1} := V_{n+1} \cap \mathrm{int}\overline{f^{-1}(U_{n+1})}$ and notes that $A_{n+1} \times U_{n+1} \subseteq O_{n+1}$ and $A_{n+1} \subseteq \mathrm{int}\overline{f^{-1}(U_{n+1})}$. With this strategy we see that:

$$\bigcap_{n\in\mathbb{N}} A_n \subseteq f^{-1}(\bigcap_{n\in\mathbb{N}} U_n) \subseteq f^{-1}(U)$$

since for any $x \in \bigcap_{n\in\mathbb{N}} A_n$ and $y \in \bigcap_{n\in\mathbb{N}} U_n \neq \emptyset$, $(x, y) \in \bigcap_{n\in\mathbb{N}}(A_n \times U_n) \subseteq \bigcap_{n\in\mathbb{N}} O_n \subseteq G(f)$, that is, $y = f(x)$ and so $x \in f^{-1}(y) \subseteq f^{-1}(\bigcap_{n\in\mathbb{N}} U_n)$. Hence $f^{-1}(U)$ is a residual subset of $\mathrm{int}\overline{f^{-1}(U)}$; which shows that f is very nearly continuous. $\square$

Remark: It would appear that the notion of "strongly Choquet" considered in this paper and elsewhere coincides with what the author in [11] calls "complete". Also "sets of interior condensation" are clearly "nearly G_δ subsets" (but not vice versa).

One might ask whether, in the absense of Baireness (of X), the previous 2 theorems remain vaild. The answer is "no". In [16] an example is given of a nearly continuous function with closed graph acting from a metric space into a complete metric space which is not continuous. Incidentally, this function provides an example of a very nearly continuous function that is not continuous, (see the proof of Theorem 2 above). In the other case it is known that for a large class of domain spaces X, if X is not a Baire space then there exists a nearly continuous mapping from X into $\mathbb{R}$ with G_δ graph that is not continuous, [6]. In fact, for metrizable spaces, this provides a characterisation of being Baire, ([11], Theorem 5). Despite this there are in fact "closed graph theorems" for functions defined on other than Baire spaces. The idea used is to replace "closed graph" by something stronger. The notion we consider here is that of a separating function, [14]. A function f acting between topological spaces X and Y is said to be *separating* if for each pair of distinct points x and y in Y there exists open neighbourhoods U of x and V of y such that $\overline{f^{-1}(U)}$ and $\overline{f^{-1}(V)}$ are separated. If f is nearly continuous then this is equivalent to saying that $\text{int}\overline{f^{-1}(U)} \cap \text{int}\overline{f^{-1}(V)} = \emptyset$. It is known that the notion of separation lies strictly between that of closed graph and that of continuity, [14]. Moreover, by making obvious modifications to the main theorem in [2] or Theorem 8 in [14] or even Theorem 3.2 in [3] we obtain the following.

Theorem 4 *Every separating and nearly continuous mapping acting from a topological space X into a partition complete space Y is continuous.*

For a homomorphism acting between topological groups the notions of "closed graph" and "separation" coincide. Hence we have that: *Every nearly continuous homomorphism with closed graph acting from a topological group G into a partition complete topological group H is continuous.* This result is not quite as good as it sounds as every partition complete topological group is Čech-complete. To see this we consider the following. First, if G is partition complete then we can construct a sequence $\{U_n : n \in \mathbb{N}\}$ of symmetric neighbourhoods of e - the identity element in G, such that (i) $U_{n+1}^2 \subseteq U_n$ for all $n \in \mathbb{N}$, (ii) each sequence $\{x_n : n \in \mathbb{N}\}$ with $x_n \in U_n$ has a cluster-point in G and (iii) $K := \bigcap_{n\in\mathbb{N}} U_n$ is a compact subgroup of G. Now, as in Theorem 1 of [1] the coset space G/K endowed with the quotient topology is metrizable and moreover the quotient mapping $g \mapsto g \cdot K$ is perfect. Since partition completeness is preserved by perfect mappings, [9] and metrizable spaces that are partition complete are Čech-complete, [9] we see that G/K is completely metrizable. The result then follows from the fact that the inverse image, under a perfect mapping, of a Čech-complete space is Čech-complete.

While on the topic of continuity of group homomorphisms, let us recall the following relevant theorem (Theorem 1) from [12]. *Let $\varphi : G \to H$ be a Souslin measurable (ie: $\varphi^{-1}(U)$ is a Souslin set in G for each open set U in H) homomorphism from a paracompact Čech-complete group G into a topological group H. Then φ is continuous.*

Note: the paracompactness hypothesis in the previous theorem is redundant as every Čech-complete group is paracompact ([1], Theorem 1).

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