EXPOSING CONDITIONS IMPLYING UNIFORMITY OF ROTUNDITY

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Abstract. If every functional which exposes a subset of the unit ball of a Banach space does so uniformly strongly (uniformly weakly) then the space is uniformly rotund (weakly uniformly rotund).

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A normed linear space X is *rotund* if every point of its unit sphere $S(X)$ is an extreme point of its closed unit ball $B(X)$. The space X is *uniformly rotund* if for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\|x - y\| < \varepsilon$ whenever $\|x + y\| \geq 2 - \delta(\varepsilon)$ and $x, y \in S(X)$. X is *weakly uniformly rotund* if for each $g \in S(X^*)$ and $\varepsilon > 0$ there exists a $\delta(\varepsilon, g) > 0$ such that $|g(x - y)| < \varepsilon$ whenever $\|x + y\| \geq 2 - \delta(\varepsilon, g)$ and $x, y \in S(X)$.

If X is uniformly rotund then for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for every $x \in S(X)$ and $f \in S(X^*)$ with $f(x) = 1$ we have that $S(B(X), f, \delta(\varepsilon)) \subseteq x + \varepsilon B(X)$ where $S(B(X), f, \delta(\varepsilon))$ denotes the *slice* $\{y \in B(X) : f(y) > 1 - \delta(\varepsilon)\}$. If X is weakly uniformly rotund then for each $g \in S(X^*)$ and $\varepsilon > 0$ there exists a $\delta(\varepsilon, g) > 0$ such that for every $x \in S(X)$ and $f \in S(X^*)$ with $f(x) = 1$ we have that $S(B(X), f, \delta(\varepsilon, g)) \subseteq x + \{y \in X : |g(y)| < \varepsilon\}$. We show that uniformity of slicing of the ball, apart from rotundity, is sufficient to imply uniform rotundity properties.

For each $f \in S(X^*)$ we will denote by $E_f \equiv \{x \in B(X) : f(x) = 1\}$ and we will say that f *exposes* $B(X)$ if $E_f \neq \emptyset$. The Bishop-Phelps Theorem guarantees that if X is a Banach space then the set of all functionals in $S(X^*)$ that expose $B(X)$ is dense in $S(X^*)$. Given an $f \in S(X^*)$ that exposes $B(X)$ we will say that E_f is *strongly exposed* by f if for each $\varepsilon > 0$ there exists a $\delta(\varepsilon)$ so that $S(B(X), f, \delta(\varepsilon)) \subseteq E_f + \varepsilon B(X)$ and that E_f is *weakly exposed* by f if for each $g \in S(X^*)$ and $\varepsilon > 0$ there exists a $\delta(\varepsilon, g) > 0$ such that $S(B(X), f, \delta(\varepsilon, g)) \subseteq E_f + \{y : |g(y)| < \varepsilon\}$. Our results are a consequence of the following general considerations.

A set-valued mapping Φ from a topological space A into subsets of the dual X^* of a normed linear space X is *weak* upper semi-continuous* $t_0 \in A$ if for each weak* open subset W of X^* such that $\Phi(t_0) \subseteq W$ there exists a neighbourhood U of t_0 such that $\Phi(U) \subseteq W$. If Φ is weak* upper semi-continuous and Φ has non-empty weak* compact convex images at each point of A then we say that Φ is a weak* *cusco* on A. Further, Φ is a *minimal weak* cusco* on A if its graph does not properly contain the graph of any other weak* cusco on A. We use the following characterisation of minimality.

Lemma 1. *([2], Lemma 2.5) A weak* cusco Φ from a topological space A into subsets of the dual X^* of a normed linear space X is a minimal weak* cusco if and only if for any non-empty open subset V of A and weak* closed convex subset K of X^*, with $\Phi(V) \not\subseteq K$, there exists a non-empty open subset $V_1 \subseteq V$ such that $\Phi(V_1) \cap K = \emptyset$.*

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A set-valued mapping Φ from a metric space (A, d) into subsets of the dual X^* of a normed linear space X is said to be *Hausdorff norm upper semi-continuous* at $t_0 \in A$ if for each $\varepsilon > 0$ there exists a $\delta(\varepsilon, t_0) > 0$ such that $\Phi(t) \subseteq \Phi(t_0) + \varepsilon B(X^*)$ for all $t \in A$ with $d(t, t_0) < \delta(\varepsilon, t_0)$ and is said to be *Hausdorff weak* upper semi-continuous* at $t_0 \in A$ if for each $x \in S(X)$ and $\varepsilon > 0$ there exists a $\delta(\varepsilon, x, t_0) > 0$ such that $\Phi(t) \subseteq \Phi(t_0) + \{f \in X^* : |f(x)| < \varepsilon\}$ for all $t \in A$ with $d(t, t_0) < \delta(\varepsilon, x, t_0)$. We will say that Φ is *uniformly Hausdorff norm upper semi-continuous* on a subset D of A if for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\Phi(s) \subseteq \Phi(t) + \varepsilon B(X^*)$ for all $s, t \in D$ with $d(s, t) < \delta(\varepsilon)$ and is said to be *uniformly Hausdorff weak* upper semi-continuous* on D if for each $x \in S(X)$ and $\varepsilon > 0$ there exists a $\delta(\varepsilon, x) > 0$ such that $\Phi(s) \subseteq \Phi(t) + \{f \in X^* : |f(x)| < \varepsilon\}$ for all $s, t \in D$ with $d(s, t) < \delta(\varepsilon, x)$. Uniformly Hausdorff upper semi-continuous mappings have significant single-valuedness properties, as shown in ([2], Proposition 3.4).

Proposition 1. *Given a metric space (A, d) and a normed linear space X, with dual X^*, a minimal weak* cusco Φ from A into subsets of X^* which is uniformly Hausdorff weak* upper semi-continuous on some dense subset D of A is single-valued on A and for each $x \in S(X)$ the mapping $t \mapsto \hat{x}(\Phi(t))$ is uniformly continuous on A. Further, if Φ is uniformly Hausdorff norm upper semi-continuous on D then Φ is single-valued and uniformly norm continuous on A.*

Proof. First we will show that Φ is single-valued on D. So let us suppose for the purpose of obtaining a contradiction that Φ is not single-valued at $t_0 \in D$. Then there exist $f_1, f_2 \in \Phi(t_0)$, $r > 0$ and $x \in S(X)$ such that $(f_1 - f_2)(x) > 3r > 0$. Consider $K \equiv \{f \in X^* : f(x) \geq f_1(x) - 2r\}$. Since Φ is uniformly Hausdorff weak* upper semi-continuous on D there exists a $\delta > 0$ so that $\Phi(s) \subseteq \Phi(t) + \{f \in X^* : |f(x)| < r\}$ whenever $s, t \in D$ and $d(s, t) < \delta$. Now, $\Phi(B(t_0, \delta)) \not\subseteq K$ since $f_2 \notin K$ so there exists a non-empty open subset V_1 of $B(t_0, \delta)$ such that $\Phi(V_1) \cap K = \emptyset$. Now for any $t \in V_1 \cap D$ we have that $f_1 \notin \Phi(t) + \{f \in X^* : |f(x)| < r\}$. But on the otherhand, $d(t_0, t) < \delta$, which means that $f_1 \in \Phi(t_0) \subseteq \Phi(t) + \{f \in X^* : |f(x)| < r\}$; which is impossible. Hence Φ is single-valued on D. For each $x \in X$ the mapping $T_x : D \to R$ defined by $T_x(t) \equiv \hat{x}(\Phi(t))$ is uniformly continuous on D and hence has a uniformly continuous extention T_x^* to A. It now follows from the weak* upper semi-continuity of Φ on A that $T_x^*(t) \in \hat{x}(\Phi(t))$ for all $t \in A$. Now, from ([4], Proposition 1.4) we have that $t \mapsto \hat{x}(\Phi(t))$ is a minimal cusco on A. Therefore for each $x \in S(X)$ the mapping $t \mapsto \hat{x}(\Phi(t)) = T_x^*(t)$ is uniformly continuous on A. In particular, this implies that Φ is single-valued on A.

In the case when Φ is uniformly Hausdorff norm upper semi-continuous on D we have from the previous argument that Φ is single-valued on A and so the mapping $\Phi_D : D \to X^*$ defined by $\Phi_D(t) \equiv \Phi(t)$ is uniformly norm continuous on D and hence has a uniformly norm continuous extension Φ_D^* to A. It now follows from the weak* continuity of Φ on A that $\Phi_D^* = \Phi$ and so Φ is uniformly norm continuous on A. □

We now relate the exposure of subsets of the unit ball of a normed linear space to continuity properties of the subdifferential mapping of the dual norm of the space. Given a normed linear space X, the *subdifferential* of the norm at $x \in X$ is the subset $\partial||x|| \equiv \{f \in B(X^*) : f(x) = ||x||\}$. The *subdifferential mapping* $x \mapsto \partial||x||$ is a weak* cusco from X into subsets of $B(X^*)$.

Lemma 2. *Let $f_0 \in S(X^*)$. If E_{f_0} is strongly exposed (weakly exposed) by $f_0 \in S(X^*)$ then the subdifferential mapping $f \mapsto \partial||f||$ from X^* into subsets of $B(X^{**})$ is Hausdorff norm upper semi-continuous (Hausdorff weak* upper semi-continuous) at f_0 and $\widehat{E_{f_0}}$ is weak* dense in $\partial||f_0||$.*

Proof. For each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$S(B(X), f, \delta(\varepsilon)/2) \subseteq S(B(X), f_0, \delta(\varepsilon)) \subseteq E_{f_0} + \varepsilon B(X)$$

for each $f \in X^*$ with $||f - f_0|| \leq \delta(\varepsilon)/2$. Hence by Goldstine's theorem we have that:

$$\begin{aligned}\partial||f|| \subseteq S(B(X^{**}), \hat{f}, \delta(\varepsilon)/2) &\subseteq \overline{S(B(\hat{X}), \hat{f}, \delta(\varepsilon)/2)}^{w^*} \\ &\subseteq \overline{S(B(\hat{X}), \hat{f}_0, \delta(\varepsilon))}^{w^*} \\ &\subseteq \overline{\widehat{E_{f_0}}}^{w^*} + \varepsilon B(X^{**}) \subseteq \partial||f_0|| + \varepsilon B(X^{**})\end{aligned}$$

for each $f \in X^*$ with $||f_0 - f|| < \delta(\varepsilon)/2$. This shows that $f \mapsto \partial||f||$ is Hausdorff norm upper semi-continuous at f_0 and that

$$\partial||f_0|| \subseteq \overline{\widehat{E_{f_0}}}^{w^*} + \varepsilon B(X^{**}) \text{ for each } \varepsilon > 0$$

which gives the first result. The proof for the case when E_{f_0} is weakly exposed by f_0 is similar, except with $\delta(\varepsilon)$ replaced by $\delta(\varepsilon, g)$, $\varepsilon B(X)$ replaced by $\{y \in X : |g(y)| < \varepsilon\}$ and $\varepsilon B(X^{**})$ replaced by $\{F \in X^{**} : |\hat{g}(F)| \leq \varepsilon\}$. □

For a normed linear space X the restriction of the subdifferential mapping $x \mapsto \partial||x||$ to $S(X)$, is a minimal weak* cusco, ([2], Lemma 3.5).

Lemma 3. *Consider a subset D of $S(X^*)$.*
(i) If for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ so that for every $f \in D$, $S(B(X), f, \delta(\varepsilon)) \subseteq E_f + \varepsilon B(X)$ then the restriction of the mapping $f \mapsto \partial||f||$ to $S(X^*)$ is uniformly Hausdorff norm upper semi-continuous on D.
(ii) If for each $g \in S(X^*)$ and $\varepsilon > 0$ there exists a $\delta(\varepsilon, g) > 0$ so that for every $f \in D$, $S(B(X), f, \delta(\varepsilon, g)) \subseteq E_f + \{y \in X : |g(y)| < \varepsilon\}$, then the restriction of the mapping $f \mapsto \partial||f||$ to $S(X^*)$ is uniformly Hausdorff weak upper semi-continuous on D.*

Proof. This follows directly from examining the proof of Lemma 2. □

By combining Proposition 1 with Lemma 3 we obtain the following geometrical consequences.

Theorem 1. *Consider a dense subset D of $S(X^*)$.*
(i) *X is uniformly rotund if for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for every $f \in D$, $S(B(X), f, \delta(\varepsilon)) \subseteq E_f + \varepsilon B(X)$.*
(ii) *X is weakly uniformly rotund if for each $g \in S(X^*)$ and $\varepsilon > 0$ there exists a $\delta(\varepsilon, g) > 0$ such that for every $f \in D$, $S(B(X), f, \delta(\varepsilon, g)) \subseteq E_f + \{y \in X : |g(y)| < \varepsilon\}$.*

Proof. In the first case we see that the restriction of the mapping $f \mapsto \partial\|f\|$ to $S(X^*)$ is single-valued and uniformly norm continuous, which implies that the dual norm is uniformly Fréchet differentiable, ([1], p.25) and which gives the result by ([1], p.134). In the second case we see that the mapping $f \mapsto \partial\|f\|$ is single-valued on $S(X^*)$ and for each $g \in S(X^*)$ the mapping $f \mapsto \hat{g}(\partial\|f\|)$ on $S(X^*)$ is uniformly continuous, which implies that the dual norm is uniformly Gâteaux differentiable, ([1], p.25) and which gives the result by ([1] p.63). □

As a further application of our theory we establish similar results for a dual space.

Theorem 2. *Consider a dense subset D of $S(X)$.*

(i) *X^* is uniformly rotund if for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for every $x \in D$, $S(B(X^*), \hat{x}, \delta(\varepsilon)) \subseteq E_{\hat{x}} + \varepsilon B(X^*)$.*

(ii) *X^* is weakly uniformly rotund if for each $G \in S(X^{**})$ and $\varepsilon > 0$ there exists a $\delta(\varepsilon, G) > 0$ such that for every $x \in D$, $S(B(X^*), \hat{x}, \delta(\varepsilon, G)) \subseteq E_{\hat{x}} + \{f \in X^* : |G(f)| < \varepsilon\}$.*

Proof. The proof of (i) follows directly from Proposition 1 and Lemma 3. For the proof of (ii), it follows from Proposition 1 and Lemma 3 that the restriction of the subdifferentiable mapping $x \mapsto \partial\|x\|$ to $S(X)$ is single-valued on $S(X)$. Hence for each $G \in S(X^{**})$ and $\varepsilon > 0$ there exists a $\delta(\varepsilon, G) > 0$ so that for every $x \in D$, $\sup\{|G(f-g)| : f, g \in S(B(X^*), \hat{x}, \delta(\varepsilon, G)) \leq 2\varepsilon$. Given $F \in S(X^{**})$ with $E_F \neq \emptyset$ consider $S(B(X^*), F, \delta(\varepsilon, G))$. For f, g any two elements of $S(B(X^*), F, \delta(\varepsilon, G))$ we have $[f, g] \cap (1 - \delta(\varepsilon, G))B(X^*) = \emptyset$. Hence, by the strong separation theorem there exists an $x \in S(X)$ so that $[f, g] \subseteq S(B(X^*), \hat{x}, \delta(\varepsilon, G))$. Since D is dense in $S(X)$ we may assume that $x \in D$ and so $|G(f-g)| \leq 2\varepsilon$; which in particular, implies that $S(B(X^*), F, \delta(\varepsilon, G)) \subseteq E_F + \{h \in X^* : |G(h)| \leq 2\varepsilon\}$. The proof now follows from Theorem 1 part (ii). □

The interesting aspect of Theorem 2 part(ii) is that it has recently been shown that there are non-reflexive Banach spaces whose dual norms are weakly uniformly rotund, [3].

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