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GAME CHARACTERIZATION OF FRAGMENTABILITY OF TOPOLOGICAL SPACES

Petar S. Kenderov[1], Warren B. Moors[2]

In the topological space X we consider a topological game such that the existence of a winning strategy for one of the players characterizes fragmentability of X. This is used to give a direct proof (avoiding any use of renorming techniques) that (X^*, weak*) is fragmentable if the Banach space X is weakly countably determined. We show also that, if on the dual unit sphere the weak and the weak* topologies coincide, then (X, weak) is fragmentable. Finally, we prove that (l^∞/c_0, weak) is not fragmentable by any metric.

1. Introduction and Game Characterization of Fragmentability

Let X be a topological space. Jayne and Rogers [2] call the space X **fragmentable** if there exists a metric $d(.,.)$ in X such that, for every $\epsilon > 0$ and every nonempty subset $A \subset X$, there exists a nonempty subset $B \subset A$ which is relatively open in A and $\operatorname{diam}(B) := \sup\{d(x,y) : x, y \in B\} \leq \epsilon$. In such a case the metric d is said to **fragment** X. This notion turned out to be useful and convenient in many situations (see [4-8], [13-15], [12]). The definition of fragmentability suggests to consider the following two players game in an arbitrary topological space X. The player Σ selects some nonempty subset A_1 of X, then the player Ω answers by selecting some nonempty subset $B_1 \subset A_1$ which is relatively open in A_1. Then again Σ selects an arbitrary nonempty subset $A_2 \subset B_1$ and, in turn, Ω picks up some nonempty relatively open subset B_2 of A_2. Repeating this alternative selection of sets the two players generate a sequence of sets

$$A_1 \supset B_1 \supset A_2 \supset B_2 \supset \dots$$

which we call a **play** and denote by $p = (A_i, B_i)_{i\geq 1}$. The player Ω is said to have won the play p if the set $\bigcap_{i\geq 1} A_i = \bigcap_{i\geq 1} B_i$ contains at most one point. Otherwise the player Σ is said to have won the play p. A **strategy** ω for the player Ω is a "rule of selection", i.e. it is a mapping which assigns to each **partial play** $A_1 \supset B_1 \supset A_2 \supset \dots \supset A_k$ some nonempty set $B_k = \omega(A_1, B_1, \dots, A_k)$ which is a relatively open

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subset of A_k. The play $p = (A_i,\ B_i)_{i\geq 1}$ is called an **ω-play** if $B_i = \omega(A_1, \ldots, A_i)$ for every $i \geq 1$. The strategy ω is said to be **winning strategy for Ω** if every ω-play is won by Ω. In what follows the game described above will be denoted by G_f.

Theorem 1.1 *The topological space X is fragmentable if, and only if, the player Ω has a winning strategy for the game G_f.*

Proof. Suppose X is fragmented by some metric d. Without loss of generality we may assume that d is bounded (otherwise we replace it by $d' := \dfrac{d}{1+d}$ which also fragments X). Because of fragmentability, for every non-empty subset $A \subset X$, there exists some non-empty relatively open subset $B \subset A$ such that $\operatorname{diam} B \leq \frac{1}{2}\operatorname{diam} A$. Define $\omega(A)$ to be such a subset B of A and define the strategy ω for Ω as a mapping (or rule of selection) which puts into correspondence to each partial play $A_1 \supset B_1 \supset \cdots \supset A_k$ the set $B_k := \omega(A_k)$. If Ω plays according to this strategy, he/she will win all ω-plays because the set $\bigcap_{i\geq 1} B_i$ will have diameter zero and so cannot contain more than one point. Note that the strategy ω defined above depends only on the last move of the player Σ.

Conversely, suppose the player Ω has a winning strategy ω for the game G_f. We will use ω to construct a **σ-relatively open partitioning** of the space X which separates the points of X (see [13]).

Put $A_1^1 := X$, $B_1^1 = \omega(A_1^1)$. If $\xi > 1$ is an ordinal such that, for $\eta < \xi$, the sets A_1^η, B_1^η, have already been defined, put $A_1^\xi := X \setminus \bigcup_{\eta<\xi} B_1^\eta$. If $A_1^\xi \neq \emptyset$, set $B_1^\xi := \omega(A_1^\xi)$. If $A_1^\xi = \emptyset$, put $\xi_1 := \xi$ and stop the process. When the latter case appears, we arrive at a family $(A_1^\xi,\ B_1^\xi)_{1\leq\xi<\xi_1}$ of partial ω-plays. Note that $\{B_1^\xi\}_{1\leq\xi<\xi_1}$ is a disjoint family which covers X and, for every $\xi < \xi_1$, the set $W_1^\xi := \bigcup_{\eta\leq\xi} B_1^\eta$ is open in X. For each ξ, $1 \leq \xi < \xi_1$, we construct now a family $(A_2^{\xi\gamma},\ B_2^{\xi\gamma})_\gamma$ of continuations (extensions) of the partial play $(A_1^\xi,\ B_1^\xi)$. Put $A_2^{\xi 1} := B_1^\xi$ and $B_2^{\xi 1} := \omega(A_1^\xi,\ B_1^\xi,\ A_2^{\xi 1})$. Suppose that, for some ordinal γ, $1 < \gamma$, all pairs $(A_2^{\xi\eta},\ B_2^{\xi\eta})_{\eta<\gamma}$ have already been defined. Put $A_2^{\xi\gamma} := B_1^\xi \setminus \bigcup_{\eta<\gamma} B_2^{\xi\eta}$. If $A_2^{\xi\gamma} \neq \emptyset$, set $B_2^{\xi\gamma} = \omega(A_1^\xi,\ B_1^\xi,\ A_2^{\xi\gamma})$. If $A^{\xi\gamma} = \emptyset$, put $\gamma_\xi := \gamma$ and finish the procedure. When the latter case occurs, we have a family $(A_1^\xi,\ B_1^\xi,\ A_2^{\xi\gamma},\ B_2^{\xi\gamma})_{1\leq\gamma<\gamma_\xi}$ of partial ω-plays which extend the partial play $(A_1^\xi,\ B_1^\xi)$. Note that $(B_2^{\xi\gamma})_{1\leq\gamma<\gamma_\xi}$ is a disjoint family which covers B_1^ξ. Furthermore, each point $x \in B_1^\xi$ uniquely identifies some $B_2^{\xi\gamma} \ni x$ and the corresponding $A_2^{\xi\gamma} \supset B_2^{\xi\gamma}$. Moreover, the sets $\bigcup_{\eta\leq\gamma} B_2^{\xi\eta}$, $1 \leq \gamma < \gamma_\xi$, are open in B_1^ξ. If the procedure is performed for every ξ, $1 \leq \xi < \xi_1$, we get the family $(A_1^\xi,\ B_1^\xi,\ A_2^{\xi\gamma},\ B_2^{\xi\gamma})_{1\leq\xi<\xi_1,\ 1\leq\gamma<\gamma_\xi}$, of partial ω-plays which, we may assume, are ordered by the lexicographical order. The disjoint family $(B_1^{\xi\gamma})_{1\leq\xi<\xi_1,\ 1\leq\gamma<\gamma_\xi}$, is "inscribed" in the disjoint family $(B_1^\xi)_{1\leq\xi<\xi_1}$ and every $x \in X$ uniquely identifies a partial ω-play $(A_1^\xi,\ B_1^\xi,\ A_2^{\xi\eta},\ B_2^{\xi\eta})$ by $x \in B_1^\xi$ and $x \in B_2^{\xi\eta}$. We will denote by Γ_1 the well ordered set $\{\xi : 1 \leq \xi < \xi_1\}$ and will consider it as a subset of the well ordered set $\Gamma_2 := \{(\xi,\gamma) : 1 \leq \xi < \xi_1, 1 \leq \gamma < \gamma_\xi\}$ (which is given the lexicographical order). Using the strategy ω we construct inductively a sequence of families of partial ω-plays $\{(A_1^\xi, B_1^\xi, A_2^\xi, \ldots, B_n^\xi)_{\xi\in\Gamma_n}\}_{n\geq 1}$ so that $(B_{n+1}^\xi)_{\xi\in\Gamma_{n+1}}$ is a disjoint cover of X which is inscribed in $(B_n^\xi)_{\xi\in\Gamma_n}$, Γ_n is a subset of Γ_{n+1} (with the order inherited from Γ_{n+1}) and each B_n^ξ, $\xi \in \Gamma_n$, is a relatively open subset of $A_n^\xi = X \setminus \bigcup_{\gamma<\xi, \gamma\in\Gamma_n} B_n^\gamma$. Note that the set $W_n^\xi := \bigcup_{\eta\leq\xi\in\Gamma_n} B_n^\eta$ is open in X. Moreover, each point $x \in X$ identifies a sequence $(B_n^{\gamma_n})_{n\geq 1}$, $\gamma_n \in \Gamma_n$, such that $x \in B_i^{\gamma_i}$, $i \geq 1$. This sequence uniquely determines, by means of the above construction, an ω-play $p(x) = (A_i^{\gamma_i}, B_i^{\gamma_i})_{i\geq 1}$. Since ω is a winning strategy for the player Ω, $\bigcap_{i\geq 1} B_i^{\gamma_i} = \{x\}$. This means that, if $y \in X$, $y \neq x$, then $y \notin B_i^{\gamma_i}$ when i is large enough. As shown by Ribarska (see [13],theorem 1.9) this suffices to deduce that X is fragmentable. To make the paper more self-contained we provide here the rest of the proof. Note that the correspondence $x \to p(x)$ defined above is one-to-one. For x', $x'' \in X$, set $p(x') = (A_i', B_i')$, $p(x'') = (A_i'', B_i'')$, where $p(x)$ is the play uniquely determined by $x \in X$. Define

$$d(x', x'') = \begin{cases} 0 & \text{if } B_i' = B_i'' \text{ for all } i \geq 1; \\ n^{-1} & \text{where } n = \min\{i : B_i' \neq B_i''\}. \end{cases}$$

It is not difficult to verify that $d(.,.)$ is a metric on X. It remains to show that $d(.,.)$ fragments X. Let $A \neq \emptyset$, $A \subset X$, and $\epsilon > 0$. Take some positive integer n_0 such that $\epsilon > \dfrac{1}{n_0}$ and put $\xi^* = \min\{\xi \in \Gamma_{n_0} : B_{n_0}^\xi \bigcap A \neq \emptyset\}$. For the open set $W := \bigcup_{\eta\leq\xi^*, \eta\in\Gamma_{n_0}} B_{n_0}^\eta$ we have $W \bigcap A = B_{n_0}^{\xi^*} \bigcap A$. This shows that $B_{n_0}^{\xi^*} \bigcap A$ is a relatively open subset of A. On the other hand, $d(x', x'') \leq \dfrac{1}{n_0} < \epsilon$ whenever x', $x'' \in B_{n_0}^{\xi^*}$. This completes the proof.

Remark 1.1 The authors are indebted to J. Orihuela (University of Murcia, Spain) for the useful discussions and suggestions with respect to the above theorem and its proof.

2. Some applications

In what follows we will show how to apply Theorem 1.1 to prove that the weak (weak*) topology on some Banach (dual Banach) spaces is fragmentable or not fragmentable. In the next two theorems which provide sufficient conditions for fragmentability our considerations rely upon an adaptation of a construction used by Christensen in [1]. In the third theorem we use an idea from a paper of Bourgain [3].

Theorem 2.1 *Let X be a Banach space with unit ball $B = \{x \in X : \|x\| \leq 1\}$. If the weak and weak* topologies in the dual Banach space X^* coincide on the unit sphere $S^* = \{x^* \in X^* : \|x^*\| = 1\}$, then $(B,$ weak$)$ is fragmentable.*

Proof. We will define a strategy ω for the player Ω which will turn out to be winning. In the course of constructing this strategy ω we will define not only the sets $B_n = \omega(A_1, B_1, \ldots, A_n)$, corresponding to the partial play $A_1 \supset B_1 \supset \cdots \supset A_n$ but also some points $x_n \in X$, $y_n \in X^*$, and some numbers d_n, so that the quadruples (B_n, x_n, y_n, d_n) satisfy certain requirements that will allow us to complete the proof.

Assume the set $A_1 \subset B$, $A_1 \neq \emptyset$, is the first choice of the player Σ. Put $x_0 = \{0\}$ and $d_1 := \inf\{t > 0 : tB \supset A_1\}$. If $d_1 = 0$, then $A_1 = \{x_0\} = \{0\}$. We put $B_1 := A_1, x_1 := x_0$ and $y_1 := 0$ in this case. Suppose $d_1 > 0$, then there exist points $x_1 \in A_1 \setminus \frac{d_1}{2}B$ and $y_1 \in X^*$, $\|y_1\| = 1$, and a number t_1 such that $\langle x_1, y_1\rangle > t_1 > \sup\langle \frac{d_1}{2}B, y_1\rangle := \sup\left\{\langle \frac{d_1}{2}b, y_1\rangle : b \in B\right\}$. The set $U_1 := \{x \in X : \langle x_1, y_1\rangle > t_1\}$ is open in (X, weak) and contains x_1. Define $\omega(A_1) = B_1$ to be the set $A_1 \bigcap U_1$. Suppose the strategy ω has already been defined for partial plays of length n together with the points $(x_i)_{i=0}^n$, $(y_i)_{i=1}^n$ and the numbers $(d_i)_{i=1}^n$ so that $x_0 = \{0\}$ and, for $i = 1, 2, \ldots, n$, the following requirements are satisfied (co $(x_0, \ldots, x_{i-1})$ stands for the convex hull of the set $\{x_0, \ldots, x_{i-1}\}$):

a) $d_i := \inf\{t > 0 : \text{co}\,(x_0, \ldots, x_{i-1}) + tB \supset A_i\}$;

b) $x_i \in A_i$; $y_i \in X^*$;

c) $\|y_i\| = 1$ if $d_i > 0$ and $y_i = 0$ if $d_i = 0$;

d) $\inf\langle B_i, y_i\rangle \geq \sup\langle \text{co}\,(x_0, \ldots, x_{i-1}) + \left(1 - \frac{1}{i+1}\right) d_i B, y_i\rangle$;

e) $\sup \|B_i - B_i\| := \text{diam} B_i \leq 2\left(d_i + \frac{1}{i+1}\right)$.

Let $A_1 \supset B_1 \supset \cdots \supset A_n \supset B_n$ be an ω-partial play for which the sequences $(d_i)_{i=1}^n$, $(x_i)_{i=0}^n$, $(y_i)_{i=1}^n$, are defined and satisfy a) – e). Suppose $A_{n+1} \neq \emptyset$, $A_{n+1} \subset B_n$, is the next choice of Σ. Put $d_{n+1} := \inf\{t > 0 : \text{co}(x_0, \ldots, x_n) + tB \supset A_{n+1}\}$. If $d_{n+1} = 0$, then $A_{n+1} \subset \text{co}(x_0, \ldots, x_n)$. The set $\text{co}(x_0, \ldots, x_n)$ is finite dimensional and the weak and norm topologies coincide in it. Therefore we can find some relatively open subset $B_{n+1} \subset A_{n+1}, B_{n+1} \neq \emptyset$, such that $\text{diam} B_i \leq \frac{2}{n+2}$. In this case we take arbitrarily some point $x_{n+1} \in A_{n+1}$ and put $y_{n+1} = 0$. Let $d_{n+1} > 0$. Then there exist points x_{n+1}, y_{n+1} such that

$$x_{n+1} \in A_{n+1} \setminus \left(\text{co}(x_0, \ldots, x_n) + \left(1 - \frac{1}{n+2}\right) d_{n+1} B\right), \quad y_{n+1} \in X^*, \ \|y_{n+1}\| = 1,$$

and a number t_{n+1} with

$$\langle x_{n+1}, y_{n+1}\rangle > t_{n+1} > \sup\langle \text{co}\,(x_0, \ldots, x_n) + \left(1 - \frac{1}{n+2}\right) d_{n+1} B, y_{n+1}\rangle.$$

The set $U_{n+1} := \{x \in X : \langle x, y_{n+1}\rangle > t_{n+1}\}$ is open in (X, weak) and contains x_{n+1}. Put $A' := U_{n+1} \bigcap A_{n+1}$. This is a relatively open, nonempty subset of A_{n+1}. Let M be a finite set such that $\text{co}\,(x_0, \ldots, x_n) \subset M + \frac{1}{n+2}B$. Such an M exists because $\text{co}\,(x_0, \ldots, x_n)$ is a norm compact set. Then $A' \subset A_{n+1} \subset \text{co}\,(x_0, \ldots, x_n) + d_{n+1}B \subset M + \left(d_{n+1} + \frac{1}{n+2}\right) B = \bigcup\left\{m + \left(d_{n+1} + \frac{1}{n+2}\right) B : m \in M\right\}$. Reducing the set

M, if necessary, we may assume that, for some $m_0 \in M$, the set

$$B_{n+1} := A' \setminus \left(\bigcup \left\{ m + \left(d_{n+1} + \frac{1}{n+2} \right) B : m \neq m_0,\ m \in M \right\} \right) \neq \emptyset.$$

Note that B_{n+1} is relatively open in A' and therefore in A_{n+1}. Also, $B_{n+1} \subset m_0 + \left(d_{n+1} + \frac{1}{n+2} \right) B$. Hence the diameter of B_{n+1} is smaller (or equal to) $2\left(d_{n+1} + \frac{1}{n+2} \right)$. Put $\omega(A_1, \ldots, B_n, A_{n+1}) := B_{n+1}$. Clearly, the requirements a) – e) are fulfilled now with $i = n+1$ as well. This completes the definition of the strategy ω. Together with every ω-play $p = (A_i, B_i)_{i \geq 1}$ we defined also the sequences $(d_i)_{i\geq 1}$, $(x_i)_{i\geq 0}$, $(y_i)_{i\geq 1}$, satisfying the requirements a) – e). We will show that $\bigcap_{i\geq 1} B_i$ contains at most one point. Note that $(d_i)_{i\geq 1}$ is a nonincreasing sequence of positive numbers. Let $d_\infty = \lim_{i\to\infty} d_i$. If x', $x'' \in \bigcap_{i\geq 1} B_i$, $x' \neq x''$, we have

$$0 < \frac{\|x' - x''\|}{2} \leq \frac{\sup \|B_i - B_i\|}{2} \leq d_i \left(1 + \frac{1}{i+1} \right)$$

for every $i \geq 1$. Thus $d_i \geq d_\infty > 0$. Let y_∞ be a weak* cluster point of $(y_i)_{i\geq 1}$ in $(X^*,$ weak*$)$. Evidently, $\|y_\infty\| \leq 1$. From properties d) and c), we obtain $\langle x' - v,\ y_p \rangle > \left(1 - \frac{1}{p+1} \right) d_p$, whenever $v \in \mathrm{co}(x_0, \ldots, x_q)$ and $1 \leq q < p$. Therefore $\langle x' - v,\ y_\infty \rangle \geq d_\infty$ for every $q \geq 1$ and $v \in \mathrm{co}\,(x_0, \ldots, x_q)$. Since $x' \in B_q \subset A_q$, property a) implies that, for some $v_q \in \mathrm{co}\,(x_0, \ldots, x_{q-1})$, $x' \in v_q + d_q B$. This yields $d_q \geq \|x' - v_q\| \geq \langle x' - v_q,\ y_\infty \rangle \geq d_\infty > 0$. Having in mind that $d_\infty = \lim_{q\to\infty} d_q$, we deduce from this that $\|y_\infty\| = 1$. By the assumptions in the theorem we see that y_∞ is not only weak* cluster point but also weak cluster point of $(y_i)_{i\geq 1}$. Now let x_∞ be some cluster point of $(x_i)_{i\geq 1}$ in $(B^{**},$ weak*$)$, where B^{**} is the unit ball of the second dual space X^{**}. By property b) we see that $x_\infty \in \overline{B_i}^*$, where $\overline{B_i}^*$ is the closure of B_i in $(B^{**},$weak$)$. It is also clear that $\inf\langle B_i,\ y_i \rangle = \min\langle \overline{B_i}^*,\ y_i \rangle$. It follows from c) and d) that $\langle x_\infty - v,\ y_p \rangle \geq \left(1 + \frac{1}{p+1} \right) d_p$, whenever $v \in \mathrm{co}\,(x_0, \ldots, x_q)$ and $q < p$. Since y_∞ is a weak cluster point of $(y_i)_{i\geq 1}$, this implies that $\langle x_\infty - v,\ y_\infty \rangle \geq d_\infty > 0$ for every $v \in \mathrm{co}\,(\bigcup_{i\geq 0}\{x_i\})$. This however contradicts the fact that x_∞ is a cluster point of $(x_i)_{i\geq 0}$ in $(B^{**},$ weak*$)$. This completes the proof.

Remark 2.1 The only role of the point x'' in the above proof was to guarantee that $d_\infty > 0$. What we actually proved concerning any ω–play $p = (A_i, B_i)_i$ was the following statement: if $\bigcap_{i\geq 1} B_i$ contains some point x', then d_∞ must be zero. In view of property e) this means that, for every ω–play $p = (A_i, B_i)_i$, $\lim_{n\to\infty} (\mathrm{diam} B_i) = 0$ whenever $\bigcap_{i\geq 1} B_i \neq \emptyset$. This property implies that the fragmenting metric $d(.,.)$ obtained in the proof of Theorem 1.1 generates a topology which is stronger than (or equal to) the norm topology on B. As shown in [9,10], this is equivalent to sigma-fragmentability of X by its norm (see [5] for the necessary definitions).

Definition 2.1 The Banach space X is called **weakly countably determined** (WCD) (see [17]), if there exists a countable family $(F_i)_{i\geq 1}$ of (not necessarily different) closed subsets of (B^{**}, weak^*) such that for every $y \in B$ there is some nested infinite subfamily $F_{i_1} \supset F_{i_2} \supset \dots$ with $y \in \bigcap_{k\geq 1} F_{i_k} \subset B$.

Note that, for such a family, if $y_k \in F_{i_k}$, $k \geq 1$, then the sequence $(y_k)_{k\geq 1}$ has a weak cluster point $y^* \in \bigcap_{k\geq 1} F_{i_k} \subset B$. In particular, $\bigcap_{k\geq 1} F_{i_k}$ is a weakly compact subset of X. Every reflexive Banach space is WCD (it suffices to take $F_i = B$ for every $i \geq 1$). More generally, every weakly compactly generated Banach space is WCD. It is also known that the class of WCD spaces is strictly larger than the class of weakly compactly generated Banach spaces [16].

The next theorem is a well known fact. It can be deduced by combining a result of Mercourakis [11] that the dual of a WCD Banach space admits an equivalent dual rotund norm and the result of Ribarska [14] that every rotund space is fragmentable. We provide here a direct proof which does not use renorming techniques.

Theorem 2.2 *Let X be a WCD Banach space. Then (X^*, weak^*) is fragmentable.*

Proof. If suffices to show that (B^*, weak^*) is fragmentable. We will show that the player Ω has a winning strategy for G_f in this space. We can assume that all F_i, $i \geq 1$, from definition 2.1 are convex and balanced sets ($F_i = -F_i$). Otherwise, one could take a new family $(F_i')_{i\geq 1}$ of sets, where F_i' is the closed absolutely convex hull of F_i in (B^{**}, weak^*). The new system $(F_i')_{i\geq 1}$ will also enjoy the property described in definition 2.1. For technical reasons, which will become obvious later, we will assume also that every set $F \in (F_i)_{i\geq 1}$ participates in this system infinitely many times (with different indexes i).

For every $n \geq 1$ we consider the semi-norm $\varphi_n : X^* \longrightarrow R$ defined by $\varphi_n(x) = \sup\{\langle y, x\rangle : y \in F_n \bigcap X\}$. Since $F_n \bigcap X \ni \{0\}$ and $F_n \subset B^{**}$, this is a well defined norm continuous function in X^*. The set $C_n := \{x \in X^* : \varphi_n(x) \leq 1\} = \{x \in X^* : \langle y, x\rangle \leq 1 \text{ for every } y \in F_n \bigcap X\}$ is weak* closed in X^* and contains the unit ball B^* of X^* (since $F_n \subset B^{**}$). By the bipolar theorem we know that $F_n \bigcap X = \{y \in B : \langle y, x\rangle \leq 1 \text{ for every } x \in C_n\}$. The construction of the strategy ω we are going to undertake now is almost identical with the construction of the strategy ω from the proof of Theorem 2.1. The main difference being that now we operate in X^* (and its elements will be denoted by x while the elements of X or X^{**} will be denoted by y) and in the definition of numbers d_i we use the seminorms φ_i instead of the norm in X^*. Let $A_1 \neq \emptyset$, $A_1 \subset B^*$ be the first choice of the player Σ. Put $x_0 = \{0\} \in X^*$ and define $d_1 := \sup\{\varphi_1(a) : a \in A_1\} = \inf\{t > 0 : x_0 + tC_1 \supset A_1\}$. Since C_1 is a norm neighbourhood of 0 in X^* and A_1 is a bounded subset of X^*, $0 \leq d_1 < \infty$. If $d_1 > 0$, there is some $x_1 \in A_1$, $x_1 \notin \frac{d_1}{2} C_1$. Since C_1 is weak* closed, we can find some $y_1 \in X$ which strictly separates x_1 from $\frac{d_1}{2}C_1$: $\langle y_1, x_1\rangle > \sup\langle y_1, \frac{d_1}{2}C_1\rangle$. Since $y_1 \neq 0$ and C_1 absorbs X^*, we may assume that $\sup\langle y_1, C_1\rangle = 1$. Then $y_1 \in F_1 \bigcap X$.

The weak* open set $U_1 := \left\{x \in X^* : \langle y_1,\ x\rangle > \sup\langle y_1,\ \frac{d_1}{2}C_1\rangle = \frac{d_1}{2}\right\}$ contains x_1. Put $B_1 = \omega(A_1) := A_1 \bigcap U_1$. This is a nonempty (containing x_1) relatively weak* open subset of A_1. Note that

$$\text{(1)} \qquad \inf\langle y_1,\ B_1\rangle \geq \frac{d_1}{2} = \sup\langle y_1,\ \frac{d_1}{2}C_1\rangle.$$

Also, since $B_1 - B_1 \subset d_1(C_1 - C_1) = 2d_1C_1$ we have

$$\text{(2)} \qquad \varphi_1(B_1 - B_1) := \sup\{\varphi_1(b' - b'') : b',\ b'' \in B_1\} \leq 2d_1 < 2(d_1 + 1)$$

If $d_1 = 0$, we put $B_1 = \omega(A_1) := A_1$ and pick up arbitrarily a point $x_1 \in A_1$ and $y_1 = \{0\} \in F_1 \bigcap X$. Hence inequality (1) is fulfilled. Moreover, since $\sup\varphi_1(A_1 - A_1) \leq 2\sup\varphi_1(A_1) = 0$, inequality (2) is also fulfilled.

Suppose the strategy ω has already been defined for all partial plays $A_1 \supset B_1 \supset \cdots \supset A_n \supset B_n$ together with some points $(x_i)_{i=0}^n$, $(y_i)_{i=1}^n$ and some numbers $(d_i)_{i=1}^n$ so that $x_0 = \{0\} \in X^*$ and, for $i = 1,\ 2,\ \ldots,\ n$,

a) $d_i = \inf\{t > 0 : \mathrm{co}(x_0,\ \ldots,\ x_{i-1}) + tC_i \supset A_i\}$;

b) $x_i \in A_i \subset X^*, y_i \in F_i \bigcap X$;

c) $\sup\langle y_i,\ C_i\rangle = 1$ if $d_i > 0$ and $y_i = \{0\}$ if $d_i = 0$;

d) $\inf\langle y_i,\ B_i\rangle \geq \sup\langle y_i,\ \mathrm{co}(x_0,\ \ldots,\ x_{i-1}) + \left(1 - \frac{1}{i+1}\right) d_iC_i\rangle$;

e) $\sup\varphi_i(B_i - B_i) \leq 2\left(d_i + \frac{1}{i}\right)$.

Let $A_1 \supset B_1 \supset \cdots \supset A_n \supset B_n$ be a partial ω-play and $A_{n+1} \neq \emptyset$, $A_{n+1} \subset B_n$, be the next choice of the player Σ. Consider the number $d_{n+1} := \inf\{t > 0 : \mathrm{co}(x_0,\ \ldots,\ x_n) + tC_{n+1} \supset A_{n+1}\}$. Clearly, $0 \leq d_{n+1} < \infty$. If $d_{n+1} > 0$, there exists some $x_{n+1} \in A_{n+1}$, $x_{n+1} \notin A := \left(\mathrm{co}(x_0,\ \ldots,\ x_n) + \left(1 - \frac{1}{n+2}\right) d_{n+1}C_{n+1}\right)$. This implies that some $y_{n+1} \in X$ strictly separates x_{n+1} from A : $\langle y_{n+1},\ x_{n+1}\rangle > \sup\langle y_{n+1},\ A\rangle \geq \sup\langle y_{n+1},\ \left(1 - \frac{1}{n+2}\right) d_{n+1}C_{n+1}\rangle > 0$. The last inequality in the chain is due to the fact that $y_{n+1} \neq 0$ and C_{n+1} is absorbing set in X^*. This means we can assume that $\sup\langle y_{n+1},\ C_{n+1}\rangle = 1$. In this case $y_{n+1} \in F_{n+1} \bigcap X$. Put $U_{n+1} := \{x \in X^* : \langle y_{n+1},\ x\rangle > \sup\langle y_{n+1},\ A\rangle = \sup\langle y_{n+1}, \mathrm{co}(x_0,\ \ldots,\ x_n)\rangle + \left(1 - \frac{1}{n+2}\right) d_{n+1}\}$ and $A' := A_{n+1} \bigcap U_{n+1}$. A' is a weak* relatively open subset of A_{n+1} containing x_{n+1}. Let M be a finite set in X^* such that $\mathrm{co}(x_0,\ \ldots,\ x_n) \subset M + \frac{1}{n+1}C_{n+1}$ (such a set M exists because $\mathrm{co}(x_0,\ \ldots,\ x_n)$ is norm compact and C_{n+1} contains B^*). Then $A' \subset A_{n+1} \subset \mathrm{co}(x_0,\ \ldots,\ x_n) + d_{n+1}C_{n+1} \subset M + \left(d_{n+1} + \frac{1}{n+1}\right) C_{n+1}$. Reducing, if necessary, the set M we can assume that $A' \subset$

$M + \left(d_{n+1} + \frac{1}{n+1}\right) C_{n+1}$ but, for some $m_0 \in M$, the set

$$B_{n+1} := A' \setminus \left(\bigcup \left\{m + \left(d_{n+1} + \frac{1}{n+1}\right) C_{n+1} : m \neq m_0,\ m \in M\right\}\right) \neq \emptyset.$$

Note that B_{n+1} is relatively open in A' and hence in A_{n+1}. Also, $B_{n+1} \subset m_0 + \left(d_{n+1} + \frac{1}{n+1}\right) C_{n+1}$ and, hence, $B_{n+1} - B_{n+1} \subset 2\left(d_{n+1} + \frac{1}{n+1}\right) C_{n+1}$. Since $B_{n+1} \subset U_{n+1}$, all requirements a) – e) are fulfilled for $i = n+1$. We define $\omega(A_1, \ldots, B_n, A_{n+1}) := B_{n+1}$ in this case.

If $d_{n+1} = 0$, put $A' = A_{n+1}$ and find some finite set M such that $\mathrm{co}(x_0, \ldots, x_n) \subset M + \frac{1}{2(n+1)} C_{n+1}$. Then $A' = A_{n+1} \subset \mathrm{co}(x_0, \ldots, x_n) + \frac{1}{2(n+1)} C_{n+1} \subset M + \frac{1}{n+1} C_{n+1} = M + \left(d_{n+1} + \frac{1}{n+1}\right) C_{n+1}$. Reasoning as above we can find some $m_0 \in M$ so that A' is not contained in $\bigcup\left\{m + \left(d_{n+1} + \frac{1}{n+1}\right) C_{n+1} : m \in M \setminus \{m_0\}\right\}$. Then the set $B_{n+1} := A' \setminus \left\{(M \setminus \{m_0\}) + \left(d_{n+1} + \frac{1}{n+1}\right) C_{n+1}\right\} \neq \emptyset$ is relatively weak* open in A'. Let x_{n+1} be any point in A_{n+1} and $y_{n+1} = \{0\} \in F_{n+1} \bigcap X$. It is easy to check that a) – d) are satisfied.

Proceeding by induction, we can think that the strategy ω is defined. We show next that this is a winning strategy for the player Ω, i.e. $\bigcap_{i \geq 1} B_i$ has at most one point. Suppose this is not the case and take $x', x'' \in \bigcap_{i \geq 1} B_i$, $x' \neq x''$. For some $y \in B$, $d^* := \langle y, x' - x'' \rangle > 0$. Since X is WCD, there is an infinite sequence $F_{i_1} \supset F_{i_2} \supset \ldots$, $i_1 < i_2 < \ldots$, for which $y \in \bigcap_{k \geq 1} F_{i_k} \subset X$. This, together with property b), implies that the sequence $(y_{i_k})_{k \geq 1}$ has a weak cluster point $y^* \in B$. On the other hand, for every $k \geq 1$, $0 < d^* = \langle y, x' - x'' \rangle \leq \varphi_{i_k}(x' - x'') \leq \sup \varphi_{i_k}(B_{i_k} - B_{i_k}) \leq 2\left(d_{i_k} + \frac{1}{i_k}\right)$. Thus $d_{i_k} \geq \frac{d^*}{2} - \frac{1}{i_k} > 0$ when k is large enough. In particular, $\sup\langle y_{i_k}, C_{i_k} \rangle = 1$ when k is big. Let x^* be a cluster point of $(x_{i_k})_{k \geq 1}$ in (B^*, weak^*). Property b) implies that $x^* \in \overline{B_{i_k}}^*$, where $\overline{B_{i_k}}^*$ is the closure of B_{i_k} in (B^*, weak^*). From d) and the fact that $\sup\langle y_{i_k}, C_{i_k} \rangle = 1$ we derive, for every $x \in \overline{B_{i_k}}^*$, $\langle y_{i_k}, x \rangle \geq \sup\langle y_{i_k}, \mathrm{co}(x_0, \ldots, x_{i_k - 1}) \rangle + \left(1 - \frac{1}{i_k + 1}\right) d_{i_k}$. For $x = x^*$ and $p > q \geq 1$ we get from this $\langle y_{i_p}, x^* - x_{i_q} \rangle \geq \left(1 - \frac{1}{i_p + 1}\right) d_{i_p}$. Letting $p \to \infty$ results in $\langle y^*, x^* - x_{i_q} \rangle \geq \frac{d^*}{2} > 0$, where $q \geq 1$ and $y^* \in B$. This inequality shows that x^* is not a weak* cluster point of $(x_{i_q})_{q \geq 1}$ which is a contradiction. This completes the proof.

Finally, we will demonstrate how to use Theorem 1.1 in order to prove that a given topological space is not fragmentable.

Consider the Banach space l^∞ of all bounded functions $x : N \to R$ with the norm $||x|| = \sup\{|x(n)| : n \in N\}$ and its closed subspace $c_0 = \{x \in l^\infty : \lim_{n\to\infty} x(n) = 0\}$. Bourgain [3] has shown that l^∞/c_0 does not admit equivalent rotund renorming. The main idea from the paper of Bourgain allows to show that the following result has place

Theorem 2.3 *(l^∞/c_0, weak) is not fragmentable.*

Remark 2.2 Having in mind the result of Ribarska [14] that the weak topology of any rotund Banach space is fragmentable, Theorem 2.3 implies the result of Bourgain that l^∞/c_0 does not admit an equivalent rotund norm.

Let $\pi : l^\infty \to l^\infty/c_0$ be the canonical mapping. By x (resp. y) we denote the elements of l^∞ (resp. l^∞/c_0). x^* and y^* will stand for the elements of the respective dual spaces $(l^\infty)^*$ and $(l^\infty/c_0)^*$. For $x \in l^\infty$, $\mathrm{supp}\,(x)$ denotes the set $\{n \in N : x(n) \neq 0\}$. We need the following auxiliary result.

Lemma 2.1 [3] *Let $x^* \in (l^\infty)^*$ and M an infinite subset of N. Then there exists an infinite subset $M' \subset M$ such that $|\langle x,\ x^*\rangle| < 1$ whenever $||x|| \le 2$ and $\mathrm{supp}\,(x) \subset M'$.*

Proof of lemma 2.1. Suppose the lemma is not true. Take some positive integer $d > 2||x^*||$ and find a disjoint family $(M_i)_{i=1}^d$ of infinite subsets of M. Since each M_i, $i = 1, \ldots, d$, fails the property stated in the lemma, there are some elements $x_i \in l^\infty$, $i = 1, 2, \ldots, d$, such that

i) $||x_i|| \le 2$;

ii) $\mathrm{supp}\,(x_i) \subset M_i$;

iii) $\langle x_i,\ x^*\rangle \ge 1$.

Then, for the vector $x = \sum_1^d x^i$, we have $||x|| \le 2$ and $\langle x,\ x^*\rangle = \sum_1^d \langle x_i,\ x^*\rangle \ge d > 2||x^*||$. This is a contradiction.

Proof of Theorem 2.3 If suffices to show that for every strategy ω of the player Ω there is at least one ω-play which is won by Σ. We will prove more. Namely, we will show that there exists a winning strategy σ for the player Σ which allows him/her to win all σ-plays. It is convenient to operate simultaneously in $X := l^\infty$ and in $Y := l^\infty/c_0$. The choices of the player Σ will be sets of the form $\pi(A)$, where A is a subset of X. The choices B of the player Ω are subsets of Y. Parallel to the construction of the strategy σ for Σ we will identify a strictly decreasing sequence $(L_i)_{i\ge1}$ of infinite subsets of N and a certain sequence $(x_i)_{i\ge1}$ of elements of X which will help us prove the theorem.

Define the strategy σ's first choice to be the set $\pi(A_1)$, where $A_1 = \{x \in X : ||x|| \le 1\}$. Let Ω's choice be some nonempty open subset B_1 of $(\pi(A_1)$, weak). Then there exists $x_1 \in A_1$, and $y_j^* \in Y^*$, $j = 1, 2, \ldots, k$, such that $\pi(x_1) \in B_1$ and $\{y \in Y : |\langle y - \pi(x_1),\ y_j^*\rangle| < 1,\ j = 1, 2, \ldots, k\} \bigcap \pi(A_1) \subset B_1$. Applying Lemma 2.1 subsequently to the functionals $x_j^* = y_j^* \circ \pi$, $j = 1, 2, \ldots, k$, we arrive at an infinite set L_1 which is a proper subset of N and is such that $|\langle z, x_j^*\rangle| < 1$ whenever $j = 1, 2, \ldots, k$, $\|z\| \leq 2$ and $\operatorname{supp}(z) \subset L_1$. Let $A_2 := \{x : \|x\| \leq 1, x(p) = x_1(p) \text{ for all } p \notin L_1\}$. For $x \in A_2$, $\operatorname{supp}(x - x_1) \subset L_1$, $\|x - x_1\| \leq 2$ and, therefore, $|\langle \pi(x) - \pi(x_1), y_j^*\rangle| = |\langle \pi(x - x_1), y_j^*\rangle = |\langle x - x_1, x_j^*\rangle| < 1$ for $j = 1, 2, \ldots, k$. This means that $\pi(A_2) \subset B_1$. Put $\sigma(\pi(A_1), B_1) := \pi(A_2)$. In general, we define inductively the strategy σ which, together with the sets $(A_i)_{i\geq 1}$, also generates the sets $(L_i)_{i\geq 1}$ and the points $(x_i)_{i\geq 1}$ so that for every σ play $\pi(A_1) \supset B_1 \supset \ldots$ the following requirements are fulfilled for $i \geq 1$:

a) L_{i+1} is an infinite proper subset of L_i;

b) $x_i \in A_i$;

c) $A_{i+1} = \{x \in X : \|x\| \leq 1, x(p) = x_i(p) \text{ for } p \notin L_i\}$.

In particular, $x_{i+1}(p) = x_i(p)$ for $p \notin L_i$. This allows us to define a point x_∞ as $x_\infty(p) = x_i(p)$ for $p \notin L_i$ and $x_\infty(p) = 1$ for $p \in \bigcap_{i\geq 1} L_i$. Evidently, $x_\infty \in \bigcap_{i\geq 1} A_i$. We will now show that $\bigcap_{i\geq 1} \pi(A_i)$ contains a point different from $\pi(x_\infty)$. Find first some infinite set $L \subset N$ such that $L \setminus L_i$ is finite for every $i \geq 1$. This is possible because $(L_i)_{i\geq 0}$ is a strictly decreasing sequence of sets. Define x'_∞ by

$$x'_\infty(p) = \begin{cases} x_\infty(p) \text{ for } p \notin L \\ x_\infty(p) + \varepsilon_p \text{ for } p \in L, \end{cases}$$

where $\varepsilon_p = -1$ if $x_\infty(p) \geq 0$ and $\varepsilon_p = 1$ if $x_\infty(p) < 0$. Clearly, $\|x'_\infty\| \leq 1$, $\operatorname{supp}(x_\infty - x'_\infty) = L$ and $x_\infty - x'_\infty \notin c_0$. Hence $\pi(x_\infty) \neq \pi(x'_\infty)$. Now we will show that $\pi(x'_\infty) \in \pi(A_i)$ for every $i \geq 1$. For $i = 1$ this is so because $x'_\infty \in A_1$. For $i \geq 2$ consider the point $x'_i \in X$ defined by

$$x'_i(p) = \begin{cases} x'_\infty(p) \text{ for } p \in L_{i-1} \\ x_\infty(p) \text{ for } p \notin L_{i-1}. \end{cases}$$

Clearly, $\|x'_i\| \leq 1$. Since $x_\infty(p) = x_{i-1}(p)$ for $p \notin L_{i-1}$, we have $x'_i(p) = x_{i-1}(p)$ for $p \notin L_{i-1}$. This means that $x'_i \in A_i$ and $\pi(x'_i) \in \pi(A_i)$. On the other hand, $x'_i(p) = x'_\infty(p)$ for $p \notin L \setminus L_{i-1}$. Indeed, for $p \in L_{i-1}$, this is seen from the definition of x'_i and, for $p \notin L \bigcup L_{i-1}$ this is clear from the definitions of x'_i and x'_∞. This means that $\sup(x'_\infty - x'_i) \subset L \setminus L_{i-1}$ and the latter is a finite set. Then $x'_\infty - x'_i \in c_0$, or $\pi(x'_\infty) = \pi(x'_i) \in \pi(A_i)$ for every $i \geq 2$. This completes the proof.

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Petar S. Kenderov
Institute of Mathematics and Informatics
Acad. G. Bonchev-Street, Block 8
1113 Sofia, Bulgaria

Warren B. Moors
The University of Auckland
Mathematics Department
Private bag 92019
Auckland, New Zealand