# An elementary proof of James' characterisation of weak compactness II

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**Abstract.** In this paper we provide an elementary proof of James' characterisation of weak compactness for Banach spaces whose dual ball is weak* sequentially compact.

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In the paper [6] the author gave a simple proof of James' theorem on weak compactness for Banach spaces whose dual ball is weak* sequentially compact. This class of spaces is quite large, because in addition to all the separable Banach spaces (whose dual ball is weak* metrisable), it contains all Asplund spaces, [5] (i.e., spaces in which every separable subspace has a separable dual space) and all spaces that admit an equivalent smooth norm, [3] (which includes all WCG spaces, [1]). In fact, it contains all Gateaux differentiability spaces, [5]. On the other hand, it does not contain $\ell_\infty(\mathbb{N})$. However, the proof in [6] still relied upon the Krein-Milman theorem, Milman's theorem and the Bishop-Phelps theorem. In this paper we obtain the same result but only rely upon the Hahn-Banach theorem and convexity. The idea of the proof comes from [7, Lemmas 4-5] and [4, Lemma 2]. For any $x$ in a normed linear space $X$ we shall define $\widehat{x} \in X^{**}$ by, $\widehat{x}(x^*) := x^*(x)$ for all $x^* \in X^*$. Then, $x \mapsto \widehat{x}$, is a linear isometric embedding of $X$ into $X^{**}$. In particular, if $X$ is a Banach space, then $\widehat{X}$ is a closed linear subspace of $X^{**}$.

Let $K$ be a weak* compact convex subset of the dual of a Banach space $X$. A subset $B$ of $K$ is called a *boundary* of $K$ if for every $\widehat{x} \in \widehat{X}$ there exists a $b^* \in B$ such that $\widehat{x}(b^*) = \sup\{\widehat{x}(y^*) : y^* \in K\}$. We shall say $B$, *$(I)$-generates* $K$, if for every countable cover $\{C_n\}_{n\in\mathbb{N}}$ of $B$ by weak* compact convex subsets of $K$, the convex hull of $\bigcup_{n\in\mathbb{N}} C_n$ is norm dense in $K$.

The main theorem relies upon the following prerequisite results.

**Lemma 1** *Let $0 < \beta$, $0 < \beta'$ and suppose that $\varphi : [0, \beta + \beta'] \to \mathbb{R}$ is a convex function. Then*

$$\frac{\varphi(\beta) - \varphi(0)}{\beta} \leq \frac{\varphi(\beta + \beta') - \varphi(\beta)}{\beta'}.$$

**Proof:** The inequality given in the statement of the lemma follows by rearranging the inequality

$$\varphi(\beta) \leq \frac{\beta}{\beta + \beta'}\varphi(\beta + \beta') + \frac{\beta'}{\beta + \beta'}\varphi(0). \qquad ☺$$

**Lemma 2** *Let $V$ be a vector space (over $\mathbb{R}$) and let $\varphi : V \to \mathbb{R}$ be a convex function. If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of nonempty convex subsets of $V$, $(\beta_n)_{n\in\mathbb{N}}$ is any sequence of strictly positive numbers, $r \in \mathbb{R}$ and*

$$\beta_1 r + \varphi(0) < \inf_{a\in A_1} \varphi(\beta_1 a),$$

*then there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in $V$ such that:*

(i) $a_n \in A_n$ *and*

(ii) $\varphi(\sum_{i=1}^{n} \beta_i a_i) + \beta_{n+1} r < \varphi(\sum_{i=1}^{n+1} \beta_i a_i)$ *for all* $n \in \mathbb{N}$.

**Proof:** We proceed in two parts. Firstly we prove that if $u \in V$ and $\beta_n r + \varphi(u) < \inf_{a \in A_n} \varphi(u + \beta_n a)$ for some $n \in \mathbb{N}$, then there exists an $a_n \in A_n$, such that

$$\beta_{n+1} r + \varphi(u + \beta_n a_n) < \inf_{a \in A_n} \varphi(u + \beta_n a_n + \beta_{n+1} a).$$

To see this, suppose that $u \in V$ and that $\beta_n r + \varphi(u) < \inf_{a \in A_n} \varphi(u + \beta_n a)$. Then there exists an $0 < \varepsilon$ such that

$$r + 2\varepsilon < \frac{\inf_{a \in A_n} \varphi(u + \beta_n a) - \varphi(u)}{\beta_n}. \qquad (*)$$

So choose $a_n \in A_n$ such that $\varphi(u + \beta_n a_n) < \inf_{a \in A_n} \varphi(u + \beta_n a) + \beta_{n+1}\varepsilon$. Let $a \in A_n$. Then $v := (\beta_n a_n + \beta_{n+1} a)/(\beta_n + \beta_{n+1}) \in A_n$ and so,

$$\begin{aligned} r + 2\varepsilon &< \frac{\varphi(u + \beta_n v) - \varphi(u + 0v)}{\beta_n} && \text{by } (*) \text{ and the fact that } v \in A_n \\ &\leq \frac{\varphi(u + (\beta_n + \beta_{n+1})v) - \varphi(u + \beta_n v)}{\beta_{n+1}} && \text{by Lemma 1.} \end{aligned}$$

Rearranging gives $\beta_{n+1}(r + \varepsilon) + [\varphi(u + \beta_n v) + \beta_{n+1}\varepsilon] < \varphi(u + \beta_n a_n + \beta_{n+1} a)$ for all $a \in A_n$. Since $\varphi(u + \beta_n a_n) < [\varphi(u + \beta_n v) + \beta_{n+1}\varepsilon]$, the desired inequality follows.

From this, we may inductively construct a sequence $(a_n)_{n \in \mathbb{N}}$. For the first step, we set $u := 0$ and then, by the hypothesis, we have that $\beta_1 r + \varphi(0) < \inf_{a \in A_1} \varphi(\beta_1 a) = \inf_{a \in A_1} \varphi(0 + \beta_1 a)$. So, by the above, there exists an $a_1 \in A_1$, such that $\beta_2 r + \varphi(\beta_1 a_1) < \inf_{a \in A_1} \varphi(\beta_1 a_1 + \beta_2 a)$.

For the $n^{\text{th}}$ step, set $u := \sum_{i=1}^{n-1} \beta_i a_i$. Since $A_n \subseteq A_{n-1}$ and by the way the $a_{n-1}$ was constructed, we have that $\beta_n r + \varphi(u) < \inf_{a \in A_{n-1}} \varphi(u + \beta_n a) \leq \inf_{a \in A_n} \varphi(u + \beta_n a)$. So, by the first result again, there exists $a_n \in A_n$, such that $\beta_{n+1} r + \varphi\left(\sum_{i=1}^{n} \beta_i a_i\right) < \inf_{a \in A_n} \varphi\left(\sum_{i=1}^{n} \beta_i a_i + \beta_{n+1} a\right)$ which completes the induction. The sequence $(a_n)_{n \in \mathbb{N}}$ has the properties claimed above. ☺

We may now state and prove the main theorem.

**Theorem 1** *Let $K$ be a weak* compact convex subset of the dual of a Banach space $X$ and let $B$ be a boundary of $K$. Then $B$, $(I)$-generates $K$.*

**Proof:** After possibly translating $K$, we may assume that $0 \in B$. Let $\{C_n\}_{n \in \mathbb{N}}$ be weak* compact, convex subsets of $K$ such that $B \subseteq \bigcup_{n \in \mathbb{N}} C_n$ and suppose, for a contradiction, that $\mathrm{co}[\bigcup_{n \in \mathbb{N}} C_n]$ is not norm dense in $K$. Then there must exist an $0 < \varepsilon$ and $y^* \in K$ such that

$$y^* \in K \backslash (\mathrm{co}[\textstyle\bigcup_{n \in \mathbb{N}} C_n] + \varepsilon B_{X^*}) \quad \text{where, } B_{X^*} := \{x^* \in X^* : \|x^*\| \leq 1\}.$$

Since, for all $n \in \mathbb{N}$, $\mathrm{co}[\bigcup_{j=1}^{n} C_j]$ is weak* compact and convex, there exist $(\widehat{x}_n)_{n \in \mathbb{N}}$ in $\widehat{X}$ such that for every $n \in \mathbb{N}$, $\|\widehat{x}_n\| = 1$ and

$$\varepsilon \leq \max\{\widehat{x}_n(x^*) : x^* \in \mathrm{co}[\textstyle\bigcup_{j=1}^{n} C_j]\} + \varepsilon = \max\{\widehat{x}_n(x^*) : x^* \in \mathrm{co}[\bigcup_{j=1}^{n} C_j] + \varepsilon B_{X^*}\} < \widehat{x}_n(y^*). \qquad (**)$$

Now, $(\widehat{x}_n(y^*))_{n\in\mathbb{N}}$ is a bounded sequence of real numbers and thus has a convergent subsequence $(\widehat{x}_{n_k}(y^*))_{k\in\mathbb{N}}$. Let $s := \lim\limits_{k\to\infty} \widehat{x}_{n_k}(y^*)$. Then, $\varepsilon \leq s$ and, after relabelling the sequence $(\widehat{x}_n)_{n\in\mathbb{N}}$ if necessary, we may assume that $|\widehat{x}_n(y^*) - s| < \varepsilon/3$ for all $n \in \mathbb{N}$. Note that this relabelling does not disturb the inequality in $(**)$.

We define $A_n := \mathrm{co}\{\widehat{x}_k : n \leq k\}$ for all $n \in \mathbb{N}$ and note that: (i) $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of nonempty convex subsets of $\widehat{X}$ and (ii) if $N < n$ and $b^* \in C_N$ then

$$g(b^*) < [g(y^*) - \varepsilon] \qquad \text{for all } g \in A_n \qquad (***)$$

since, $\{\widehat{x}_k : n \leq k\} \subseteq \{\widehat{x} \in \widehat{X} : \widehat{x}(b^* - y^*) < -\varepsilon\}$; which is convex. Next, we define $p : \widehat{X} \to \mathbb{R}$ by,

$$p(\widehat{x}) := \sup_{x^*\in K} \widehat{x}(x^*) \quad \text{ for all } \widehat{x} \in \widehat{X}.$$

Then $p$ defines a convex functional on $\widehat{X}$ such that $p(0) = 0$. Moreover, for all $g \in A_1$, we have $(s-\varepsilon/3) < g(y^*) \leq p(g)$ since $\{\widehat{x}_n\}_{n\in\mathbb{N}} \subseteq \{\widehat{x} \in \widehat{X} : (s-\varepsilon/3) < \widehat{x}(y^*)\}$; which is convex and $y^* \in K$.

Let $(\beta_n)_{n\in\mathbb{N}}$ be any sequence of positive numbers such that $\lim_{n\to\infty} \left(\sum_{i=n+1}^{\infty} \beta_i\right)/\beta_n = 0$. Now, $\beta_1(s-\varepsilon/2) + p(0) < \beta_1(s-\varepsilon/3) \leq \beta_1[\inf_{g\in A_1} p(g)] = \inf_{g\in A_1} p(\beta_1 g)$.

Therefore, by Lemma 2, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in $\widehat{X}$ such that $g_n \in A_n$ and

$$p(\textstyle\sum_{i=1}^{n}\beta_i g_i) + \beta_{n+1}(s-\varepsilon/2) < p(\sum_{i=1}^{n+1}\beta_i g_i) \quad \text{ for all } n \in \mathbb{N}.$$

Since $\|g_n\| \leq 1$ for all $n \in \mathbb{N}$, we have that $\sum_{i=1}^{\infty} \|\beta_i g_i\| \leq \sum_{i=1}^{\infty} \beta_i < \infty$. As $X$ is a Banach space, this implies that $g := \sum_{i=1}^{\infty} \beta_i g_i \in \widehat{X}$ and so there exists a $b^* \in B$ such that $p(g) = g(b^*)$. Then,

$$\begin{aligned} \beta_n(s-\varepsilon/2) &< p(\textstyle\sum_{i=1}^{n}\beta_i g_i) - p\Big(\sum_{i=1}^{n-1}\beta_i g_i\Big) \leq p(g) - p\Big(\sum_{i=1}^{n-1}\beta_i g_i\Big) \\ &\leq g(b^*) - \textstyle\sum_{i=1}^{n-1}\beta_i g_i(b^*) = \sum_{i=n}^{\infty}\beta_i g_i(b^*). \end{aligned}$$

Since $B \subseteq \bigcup_{n\in\mathbb{N}} C_n$, $b^* \in C_N$ for some $N \in \mathbb{N}$. Thus, if $N < n$, then

$$(s-\varepsilon/2) < \frac{1}{\beta_n}\Big(\textstyle\sum_{i=n+1}^{\infty}\beta_i g_i(b^*)\Big) + g_n(b^*) < \frac{1}{\beta_n}\Big(\sum_{i=n+1}^{\infty}\beta_i g_i(b^*)\Big) + [g_n(y^*) - \varepsilon] \quad \text{by } (***),$$

since $g_n \in A_n$. By taking the limit as $n$ tends to infinity we get that $(s-\varepsilon/2) \leq (s-\varepsilon)$; which is impossible. Therefore, $B$, $(I)$-generates $K$. ☺

**Remark 1** *If* $\beta_n := \dfrac{1}{n!}$ *for all* $n \in \mathbb{N}$ *or,* $\beta_n := \dfrac{1}{2^{n^2}}$ *for all* $n \in \mathbb{N}$*, then* $\lim\limits_{n\to\infty} \dfrac{\sum_{i=n+1}^{\infty}\beta_i}{\beta_n} = 0$.

We will say that a subset $C$ of a Banach space $X$ is *weakly compactly generated* if for every $0 < \varepsilon$ there exists a countable family $\{C_n^{\varepsilon}\}_{n\in\mathbb{N}}$ of weakly compact convex subsets of $X$ such that $C \subseteq [\bigcup_{n\in\mathbb{N}} C_n^{\varepsilon}] + \varepsilon B_X$. Here, $B_X$ denotes the closed unit ball in the Banach space $X$. Our first compactness result is based upon the following observation: For each $\mathscr{F} \in X^{***}$ there exists an $x^* \in X^*$ such that $\mathscr{F}|_{\widehat{X}} = \widehat{x^*}|_{\widehat{X}}$. In this way we see that the relative weak topology on $\widehat{X}$ coincides with the relative weak$^*$ topology on $\widehat{X}$. In particular, each weak$^*$ compact subset of $\widehat{X}$ is weakly compact (and of course, vice versa).

**Corollary 1** *Let $C$ be a closed and bounded convex subset of a Banach space $X$. If $C$ is weakly compactly generated and every continuous linear functional on $X$ attains its supremum over $C$, then $C$ is weakly compact.*

**Proof:** Let $K := \overline{\widehat{C}}^{w^*}$. To show that $C$ is weakly compact it is sufficient to show that for every $0 < \varepsilon$, $K \subseteq \widehat{X} + 2\varepsilon B_{X^{**}}$. To this end, fix $0 < \varepsilon$ and let $\{C_n^\varepsilon\}_{n\in\mathbb{N}}$ be any countable family of weakly compact convex subsets of $X$ such that $C \subseteq [\bigcup_{n\in\mathbb{N}} C_n^\varepsilon] + \varepsilon B_X$. For each $n \in \mathbb{N}$, let $K_n^\varepsilon := K \cap [\widehat{C_n^\varepsilon} + \varepsilon B_{X^{**}}]$. Then $\{K_n^\varepsilon\}_{n\in\mathbb{N}}$ is a cover of $\widehat{C}$ by weak* closed convex subsets of $K$. Since $\widehat{C}$ is a boundary of $K$, $K \subseteq \overline{\text{co}} \bigcup_{n\in\mathbb{N}} K_n^\varepsilon \subseteq \widehat{X} + 2\varepsilon B_{X^{**}}$. ☺

The author in [6] used Theorem 1 to give a short proof of the following result.

**Corollary 2 ([6, Theorem 3])** *Let $C$ be a closed and bounded convex subset of a Banach space $X$. If $(B_{X^*}, weak^*)$ is sequentially compact and every continuous linear functional on $X$ attains its supremum over $C$, then $C$ is weakly compact.*

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