Maths 260 Lecture 17

- **Topics for today:**
  Linear systems
- **Reading for this lecture:** BDH Section 3.1, 3.2
- **Suggested exercises:**
  BDH Section 3.1 #5, 7, 9, 24, 27, 29
  BDH Section 3.2 #1, 5, 11, 13, 25
- **Reading for next lecture:**
  BDH Section 3.3
- **Today's handouts**: Exercises using complex numbers

Linear Systems

- **Linear systems** are an important class of systems of DEs, because some important physical models are linear but also because we can use linear systems to help understand nonlinear systems.
- A **linear system** is a system of DEs where the dependent variables only appear to the first power.

Matrix form

- Linear systems can be written as

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where **Y** is a vector and **A** is a matrix of constants:

$$\mathbf{Y} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix}$$

- **A** is called the coefficient matrix. The number of dependent variables is called the dimension of the system.

Example 1: Rewrite the system

$$\frac{dx}{dt} = 2x - z$$
$$\frac{dy}{dt} = -x - z$$
$$\frac{dz}{dt} = x + y$$

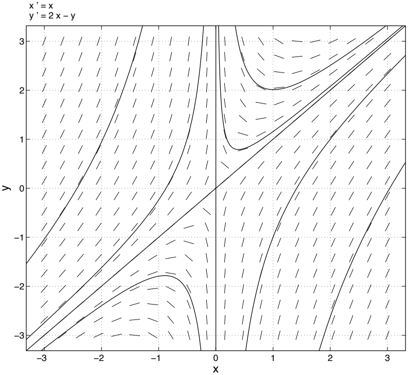
in matrix form.

Example from the last lecture

Example 2:

$$\dot{x} = x, \qquad \dot{y} = 2x - y$$

- The direction field and some solutions from *pplane*:



Example from the last lecture

Example 2:

$$\dot{x} = x, \qquad \dot{y} = 2x - y$$

- We can write this system as

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where

$$\mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}.$$

Example from the last lecture

- This system decoupled and we could find the analytic solution. We found that

$$\mathbf{Y}_1(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}, \quad \mathbf{Y}_2(t) = \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix}$$

are the (straight-line) solutions of this system and that all solutions can be written as

$$\mathbf{Y}(t) = \begin{pmatrix} c_1 e^t \\ c_1 e^t + c_2 e^{-t} \end{pmatrix} = c_1 \mathbf{Y_1} + c_2 \mathbf{Y_2},$$

i.e., as a linear combination of the straight-line solutions $\mathbf{Y}_1$ and $\mathbf{Y}_2$.

Some properties of linear systems

- Equilibrium solutions of
$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$
are values of $\mathbf{Y_0}$ such that $\mathbf{AY_0} = \mathbf{0}$.

- From linear algebra, know that if $\det(\mathbf{A}) \neq 0$, then the only solution of $\mathbf{AY_0} = \mathbf{0}$ is $\mathbf{Y_0} = \mathbf{0}$ (called the trivial solution).

- Thus, if $\det(\mathbf{A}) \neq 0$, then $\mathbf{Y}(t) = 0$ is the only equilibrium solution to
$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

Finding straight-line solutions

- At a point (x, y) on a straight-line solution, the vector field must point in the same (or opposite) direction as the vector from the origin to (x, y).
- This means

$$\mathbf{Av} = \lambda \mathbf{v} \tag{1}$$

  where $\mathbf{v} = (x, y)$ and λ is a real number.

  - If $\lambda > 0$, the vector field points in same direction as $\mathbf{v}$, i.e., away from the origin.
  - If $\lambda < 0$, the vector field points in opposite direction to $\mathbf{v}$, i.e., towards the origin.
- A number λ that satisifies Equation (1) (for non-zero $\mathbf{v}$) is called an *eigenvalue* of $\mathbf{A}$. The vector $\mathbf{v}$ is called an *eigenvector* with corresponding eigenvalue λ.

Finding straight-line solutions

- We can show that if $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ with corresponding eigenvalue λ, then

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{v}$$

is a straight-line solution to

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}.$$

- As t varies, $e^{\lambda t}$ just increases or decreases or remains constant (depending on λ) and $\mathbf{v}$ is constant, so the solution curve for $\mathbf{Y}(t)$ is a straight line.

Example 2 again: Find any straight-line solutions to

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

Linearity Principle

- If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are both solutions to

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

then so is

$$k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$$

for any constants k_1 and k_2.

- The function

$$k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$$

is called a **linear combination** of $\mathbf{Y_1}$ and $\mathbf{Y_2}$.

Linear independence of vectors in the plane

- Two vectors in the plane are linearly independent if neither vector is a multiple of the other, i.e., if they do not both lie on the same line through the origin.
- e.g. $v_1 = (1, 1)$, $v_2 = (2, -1)$ are linearly independent.
- e.g. $v_1 = (1, 1)$ and $v_3 = (-2, -2)$ are linearly dependent.

Linear independence of vectors in the plane

- If two vectors (x_1, y_1) and (x_2, y_2) are linearly independent, then for any other planar vector (x_0, y_0) there are constants k_1 and k_2 such that

$$k_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + k_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Linear independence of solutions to a 2D system

- Consider the DE
$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$
where $\mathbf{A}$ is a 2×2 matrix.
- If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions with $\mathbf{Y}_1(0)$ and $\mathbf{Y}_2(0)$ linearly independent vectors, then $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are linearly independent vectors for all t.
- We say that $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are linearly independent solutions.
- If $\mathbf{Y}(0)$ is some initial condition, then every solution to the IVP
$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}, \quad \mathbf{Y}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$
can be expressed as a linear combination of $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$.

Linear independence of solutions to a 2D system

- That is, we can write the solution to

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

as

$$\mathbf{Y}(t) = c_1\mathbf{Y}_1(t) + c_2\mathbf{Y}_2(t)$$

for appropriately chosen c_1 and c_2.

- We can find linearly independent solutions by finding the eigenvectors of **A**.

The general solution

- Let λ_1 and λ_2 be two real and distinct eigenvalues for the matrix $\mathbf{A}$, with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

- Hence, the two straight-line solutions

$$\mathbf{Y}_1(t) = e^{\lambda_1 t}\mathbf{v}_1 \quad \text{and} \quad \mathbf{Y}_2(t) = e^{\lambda_2 t}\mathbf{v}_2$$

  are linearly independent at $t = 0$ and thus are linearly independent solutions for all t.

- And we can write the general solution as

$$\mathbf{Y}(t) = c_1\mathbf{Y}_1(t) + c_2\mathbf{Y}_2(t)$$

Linear independence of vectors in higher dimensions

- These results can be generalised to higher dimensions.
- A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is linearly dependent if there are constants $c_1, c_2, \ldots, c_m$ (not all zero) such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m = \mathbf{0} \tag{2}$$

- If all the constants c_i are zero whenever equation (2) is satisfied, the set of vectors is linearly independent.

Main result

- If $\mathbf{Y}_1(t)$, $\mathbf{Y}_2(t)$,..., $\mathbf{Y}_m(t)$, are linearly independent solution vectors to the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where $\mathbf{A}$ is an $m \times m$ matrix, then the general solution to the system is

$$\mathbf{Y}(t) = c_1\mathbf{Y}_1(t) + c_2\mathbf{Y}_2(t) + \cdots + c_m\mathbf{Y}_m(t)$$

where c_1, c_2,..., c_m are arbitrary constants. That is, every solution to the system can be written in this form by appropriate choice of c_1, c_2,..., c_m.

Finding linearly independent solutions

- If a matrix $\mathbf{A}$ has distinct eigenvalues λ_j then the eigenvectors $\mathbf{v_j}$ are linearly independent.

Example 3: Find three linearly independent solutions to

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 3 & -4 & 0 \\ 0 & 1 & -2 \end{pmatrix}.$$

Hence find the solution to the IVP

$$\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}.$$

Grand summary

If $\mathbf{A}$ is an $m \times m$ matrix with real eigenvalues $\lambda_1, ..., \lambda_k$, with corresponding eigenvectors $\mathbf{v}_1, ..., \mathbf{v}_k$, then

$$\mathbf{Y}_1 = e^{\lambda_1 t}\mathbf{v}_1, \ \ldots, \mathbf{Y}_k = e^{\lambda_k t}\mathbf{v}_k$$

are straight-line solutions of the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}.$$

Furthermore, if all the λ_i are distinct and $k = m$ (i.e., there are m real and distinct eigenvalues of $\mathbf{A}$), then the set $\{\mathbf{Y}_1, ..., \mathbf{Y}_k\}$ is linearly independent and the general solution to the system is

$$\mathbf{Y}(t) = c_1\mathbf{Y}_1 + ... + c_m\mathbf{Y}_m.$$

Important ideas from today's lecture:

- Straight line solutions
- How to write linear systems in matrix form
- If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are both solutions to $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$,

  then so is $k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$ for any constants k_1 and k_2.
- If $\lambda_1, \ldots, \lambda_m$ are distinct real eigenvalues of $\mathbf{A}$ with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$, then

  $$\mathbf{Y}_1(t) = e^{\lambda_1 t}\mathbf{v}_1,\ \ldots, \mathbf{Y}_m(t) = e^{\lambda_m t}\mathbf{v}_m$$

  are straight-line solutions of the system

  $$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

  and the general solution is

  $$\mathbf{Y}(t) = c_1\mathbf{Y}_1(t) + c_2\mathbf{Y}_2(t) + \ldots + c_m\mathbf{Y}_m(t)$$

  for constants $c_1, c_2, \ldots, c_m$.