Maths 260 2005S. TEST.

Name ______________________ I.D. Number ______________________

1. (10 marks)

(a) Find the solution to the following initial value problem
(5)

$$\frac{dy}{dt} = 2t(y+1), \quad y(0) = 2.$$

$$\int \frac{dy}{y+1} = \int 2t\,dt \qquad y+1 \neq 0$$

$$\ln|y+1| = t^2 + c$$

$$|y+1| = e^c e^{t^2}$$

$$y+1 = ke^{t^2} \qquad k \in \mathbb{R}$$

(we get $k=0$ by noting that $y=-1$ is a soln even though it does not come from this method of solution).

$$y = -1 + ke^{t^2}$$

$$y(0) = 2$$

$$2 = -1 + k$$

$$k = 3$$

$$y(t) = -1 + 3e^{t^2}$$

(b) Find the general solution to the differential equation
(5)

$$\frac{dy}{dt} = y + e^t.$$

$$\frac{dy}{dt} - y = e^t$$

$$\mu(t) = \exp\left(\int(-1)\,dt\right)$$
$$= e^{-t}$$

$$e^{-t}\frac{dy}{dt} - ye^{-t} = 1$$

$$\frac{d}{dt}\left(ye^{-t}\right) = 1$$

$$ye^{-t} = t + c$$

$$y = te^t + ce^t$$

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2. (~~8~~ 10 marks)
Consider the initial value problem

$$\frac{dy}{dt} = y + t, \quad y(1) = 3.$$

(a) Use 2 steps of the Improved Euler method to approximate $y(1.2)$. Keep 6 decimal places.

(b) The 16 steps of the fourth order Runge-Kutta was used to estimate $y(1.2)$. The result was 3.907014. Estimate the error in your approximation in (a).

(c) Estimate the error if you had used 4 steps instead of 2 steps in (a). Give reasons for your answer.

(a) $h = 0.1, \quad t_0 = 1, \quad y_0 = 3$

$m_1 = f(1, 3) = 4$

$m_2 = f(1.1, 3 + 0.1 \times 4) = f(1.1, 3.4) = 4.5$

$y_1 = 3 + \frac{.1}{2}(4 + 4.5) = 3.425$

$m_1 = f(1.1, 3.425) = 4.525$

$m_2 = f(1.2, 3.425 + 0.1 \times 4.525) = f(1.2, 3.8775) = 5.0775$

$y_2 = 3.425 + \frac{.1}{2}(4.525 + 5.0775) = 3.905125$

(b) Error $\approx |3.907014 - 3.905125| = 1.9 \times 10^{-3}$

(c) $h \to \frac{h}{2}, \quad E\left(\frac{h}{2}\right) = C\left(\frac{h}{2}\right)^2 = \frac{Ch^2}{4} = \frac{E(h)}{4}$

$\therefore$ error will be approx. $\frac{1.9 \times 10^{-3}}{4} \approx 5 \times 10^{-4}$

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3. (~~10~~ 18 marks)

Consider the following differential equation:

$$\frac{dy}{dt} = y^2 - 2y - \mu$$

(a) Find all equilibrium solutions and determine their types (e.g., sink, node).

(b) For $\mu = 0$ draw the phase line, and sketch the solutions. There is no need to find explicit solutions.

(c) Draw the bifurcation diagram. Identify any values of μ for which a bifurcation exists.

(a) $y^2 - 2y - \mu = 0$

$y = \frac{2 \pm \sqrt{4 + 4\mu}}{2} = 1 \pm \sqrt{1+\mu}$.

For $\mu > -1$, 2 solns at $1 \pm \sqrt{1+\mu}$

$\mu = -1$, 1 soln at 1

$\mu < -1$, no solns.

$\frac{\partial f}{\partial y} = 2y - 2$

$\left.\frac{\partial f}{\partial y}\right|_{1+\sqrt{1+\mu}} = \sqrt{1+\mu} > 0 \Rightarrow$ source

$\left.\frac{\partial f}{\partial y}\right|_{1-\sqrt{1+\mu}} = -\sqrt{1+\mu} < 0 \Rightarrow$ sink.

$\left.\frac{\partial f}{\partial y}\right|_{1} = 0 \Rightarrow$ no info from linearisation

For $\mu = -1$, $f(y) = y^2 - 2y + 1 = (y-1)^2 > 0$ for $y \neq 1$.

$\therefore$ $y = 1$ will be a node.

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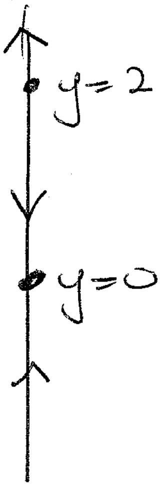
(b) For $\mu=0$

$$\frac{dy}{dt} = y^2-2y = y(y-2)$$

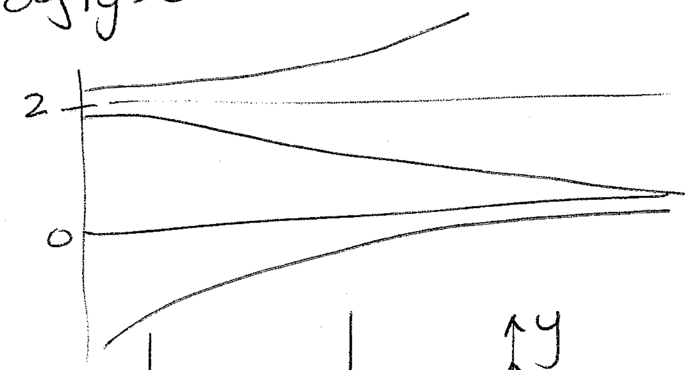
Eq at $y=0$, $y=2$



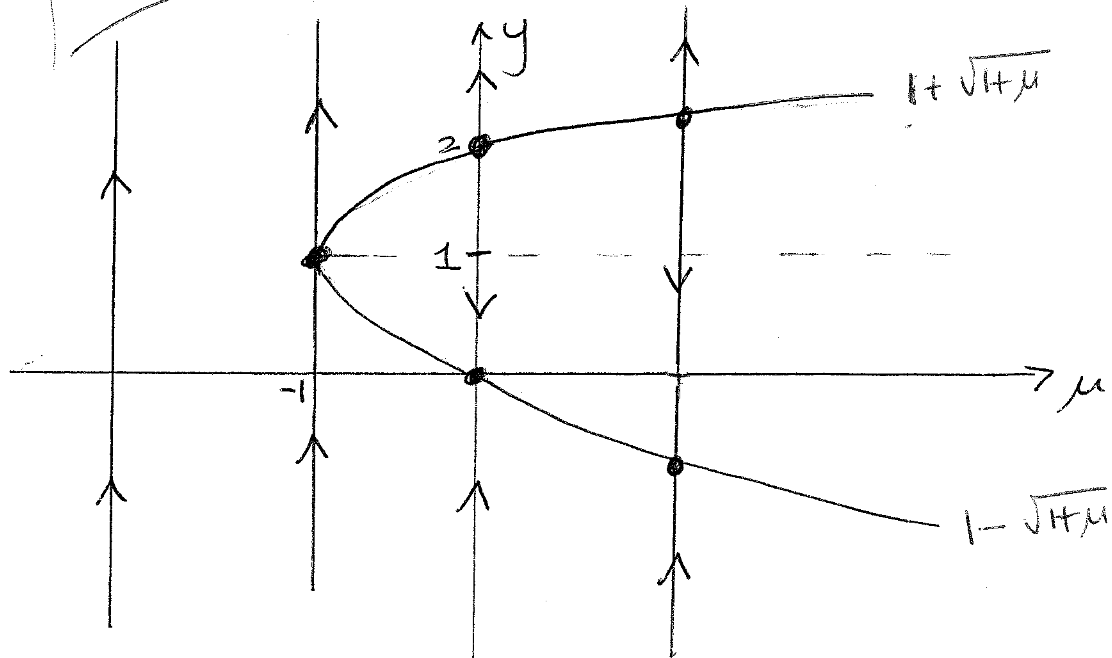
$$\frac{\partial f}{\partial y} = 2y-2$$

$$\left.\frac{\partial f}{\partial y}\right|_{y=2} = 2 > 0 \quad \text{source}$$

$$\left.\frac{\partial f}{\partial y}\right|_{y=0} = -2 < 0 \quad \text{sink}$$



(c)



For $\mu < -1$, $f(y) = y^2 - 2y - \mu$

$$= (y-1)^2 - 1 - \mu > 0 \quad \text{since } -1-\mu > 0.$$

Bifurcation at $\mu = -1$.

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4. (~~10~~ 12 marks)

Consider the following system of equations

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix} \mathbf{Y}.$$

(a) Find the general real solution to this system of equations.

(b) Sketch the phase portrait.

(a) $\det\begin{pmatrix} 2-\lambda & -1 \\ 2 & -\lambda \end{pmatrix} = -\lambda(2-\lambda)+2$

$= \lambda^2 - 2\lambda + 2.$

$\lambda = \dfrac{2 \pm \sqrt{4-8}}{2}$

$= 1 \pm i$

Eigenvector for $\lambda = 1+i$

$$\begin{pmatrix} 2-1-i & -1 \\ 2 & -1-i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-i & -1 \\ 2 & -1-i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$(1-i)x - y = 0$ Choose $\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$

Complex soln.

$$e^{(1+i)t}\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

$$= e^t(\cos t + i\sin t)\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

$$= e^t\begin{pmatrix} \cos t \\ \cos t + \sin t \end{pmatrix} + ie^t\begin{pmatrix} \sin t \\ -\cos t + \sin t \end{pmatrix}$$

General soln

$$Y(t) = C_1 e^t\begin{pmatrix} \cos t \\ \cos t + \sin t \end{pmatrix} + C_2 e^t\begin{pmatrix} \sin t \\ -\cos t + \sin t \end{pmatrix}$$

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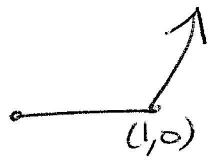
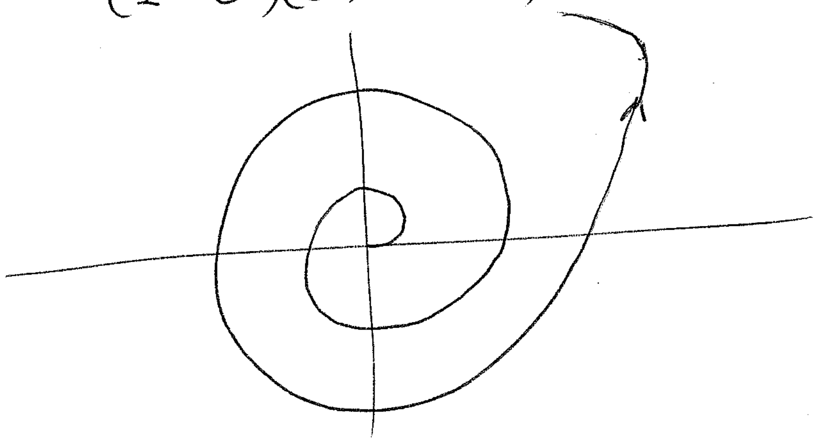
(b) Since $Re(\lambda) = 1 > 0$ it will be a spiral source.

$$\begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$





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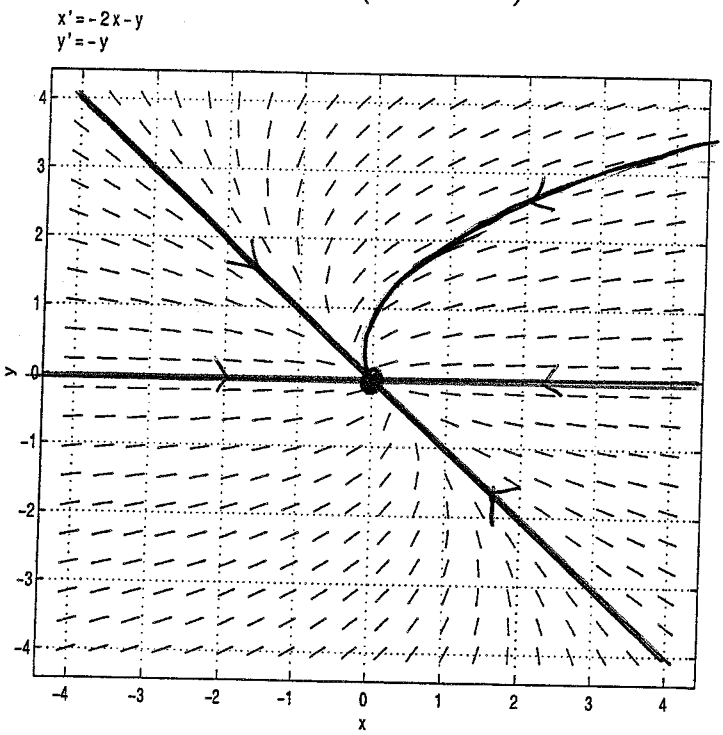
5. (10 marks)

The phase portrait for the following system of differential equations is shown below.

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & -1 \\ 0 & -1 \end{pmatrix} \mathbf{Y}.$$



equilibrium at the origin

(a) Find the straight line solutions.

(b) On this picture:

i. show all equilibrium solutions;

ii. draw the straight solutions;

iii. draw a solution which satisfies $\mathbf{Y}(0) = (1, 2)^T$.

For each solution curve you draw, you should draw an arrow indicating the direction moved as time increases.

(a) Eigenvalues -2 & -1

Eigenvector $\lambda=-2$ $\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow y=0$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda=-1$ $\begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow y=-x$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Straight line solns are $c_1 e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $c_2 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

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