Maths 260 Lecture 15

- **Topics for today:**
  Numerical methods for systems
  Existence and Uniqueness Theorem for systems
- **Reading for this lecture:** BDH Section 2.4
- **Suggested exercises:** BDH Section 2.4, #7,8,9,10
- **Reading for next lecture:** BDH Section 2.3
- **Today's handouts**: none

Numerical Methods for Systems

- Numerical methods used for first order equations can be generalised to systems of first order equations.

Example: Euler's Method for systems

- Given the IVP
$$\begin{aligned} \frac{dx}{dt} &= f(t, x, y), \\ \frac{dy}{dt} &= g(t, x, y), \end{aligned}$$
with $x(t_0) = x_0$ and $y(t_0) = y_0$, then Euler's Method calculates the approximate solution at $t_1 = t_0 + h$ to be
$$\begin{aligned} x(t_1) &\approx x_0 + hf(t_0, x_0, y_0), \\ y(t_1) &\approx y_0 + hg(t_0, x_0, y_0). \end{aligned}$$
- The process can be repeated to find an approximation after n steps.

Example 1: Use Euler's method with $h = 0.1$ to calculate an approximate solution at $t = 0.2$ to the IVP

$$\begin{aligned} \frac{dx}{dt} &= t + y, \\ \frac{dy}{dt} &= x - y^2 \end{aligned}$$

where $x(0) = 1,\ y(0) = 0.$

The results in tabular form are:

n	t_n	x_n	y_n	$f(t_n, x_n, y_n)$	$g(t_n, x_n, y_n)$
0	0	1	0	0	1
1	0.1	1	0.1	0.2	0.99
2	0.2	1.02	0.199		

Vector Form of Euler's Method

- Write

$$\mathbf{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \quad \mathbf{Y}_0 = \begin{pmatrix} y_1(t_0) \\ y_2(t_0) \\ \vdots \\ y_n(t_0) \end{pmatrix},$$

and

$$\mathbf{F}(t, \mathbf{Y}) = \begin{pmatrix} f_1(t, y_1, y_2, \ldots, y_n) \\ f_2(t, y_1, y_2, \ldots, y_n) \\ \vdots \\ f_n(t, y_1, y_2, \ldots, y_n) \end{pmatrix}.$$

- Then the Euler approximation to the solution of the IVP

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(t, \mathbf{Y}), \quad \mathbf{Y}(t_0) = \mathbf{Y}_0$$

at $t_1 = t_0 + h$ is

$$\mathbf{Y}(t_1) \approx \mathbf{Y}_0 + h\mathbf{F}(t_0, \mathbf{Y}_0)$$

Errors in numerical methods

- It can be proved that Euler's method for systems is first order, i.e., the error in the ith component of $\mathbf{Y}$ is

$$|E_i(h)| \approx k_i h$$

  in the limit of small h, where k_i is a constant.

- Thus, halving the step size will approximately halve the error in the estimated value of each component of $\mathbf{Y}$.

- Improved Euler and the 4th order Runge-Kutta methods also generalise to systems, and are of order 2 and 4 respectively.

Practical issues in using numerical methods for systems

- As for single DEs, RK4 is the most commonly used fixed stepsize numerical method. It is easy to implement and of high order.
- With this method, it is important to check whether any given stepsize is small enough to give good accuracy. An easy way to do this is to halve the stepsize and repeat the computation to see if it makes a difference to the answer.

Example 2: In *pplane*, using RK4 with default settings on the default system gives inaccurate results, as can be seen by comparing the phase portrait with that obtained with $1/10^{\text{th}}$ of the stepsize.

- Fixed stepsize methods such as RK4 may be unsuitable for
  - so-called **stiff** systems (in which there are two or more very different time scales in the problem);
  - Hamiltonian systems (in which energy is conserved);
  - very long computations, or those in which high accuracy is required.
- Another option with *pplane* is to use a Dormand-Prince method suitable for systems. This method is variable stepsize, order 5, and uses splines for fitting.
- Be careful with all numerical methods, as all methods can give misleading results under some circumstances. Always think critically about numerical results – ask yourself if the numerical results fit in with your intuition or with results you have from other methods. If not, work out why not instead of blindly trusting the numerical results.

Existence and Uniqueness Theorem for systems

- Consider the IVP

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(t, \mathbf{Y}), \qquad \mathbf{Y}(t_0) = \mathbf{Y}_0.$$

- If $\mathbf{F}$ is continuous and has continuous first partial derivatives with respect to all the dependent variables, then there is a constant $\epsilon > 0$ and a function $\mathbf{Y}(t)$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$ such that $\mathbf{Y}(t)$ is a solution to the IVP.
- For t in this interval, the solution is unique.

Interpretation of the E & U Theorem

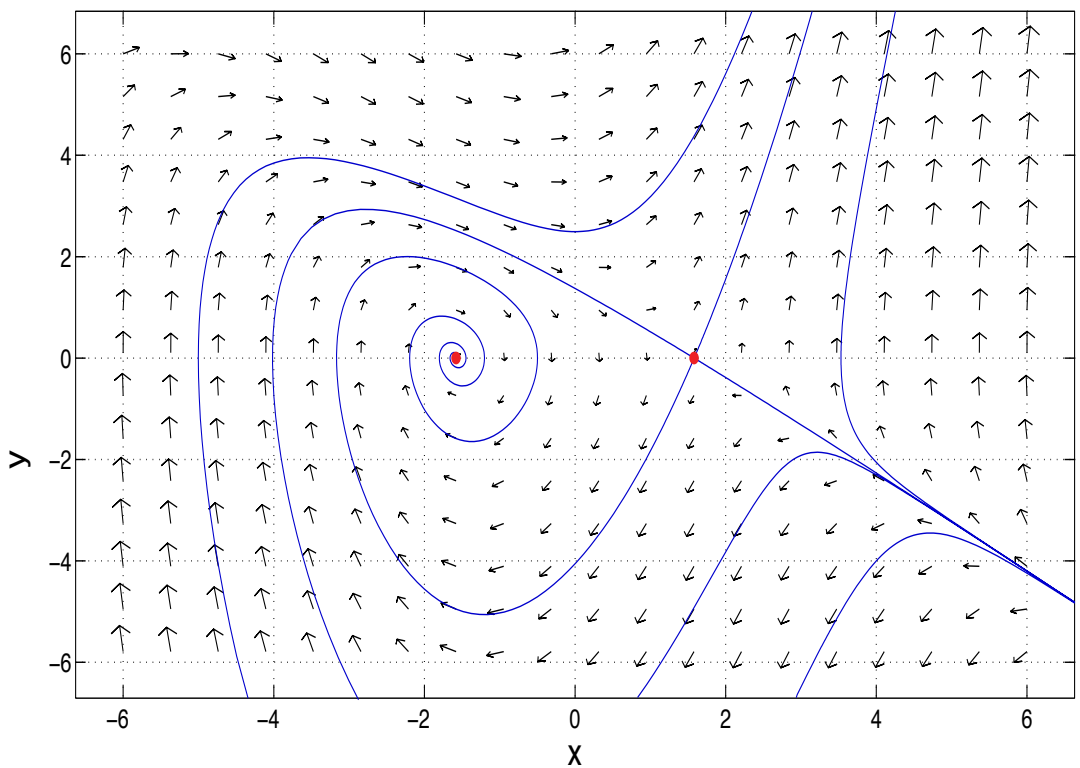
- If a system of equations is 'nice' enough, an IVP is guaranteed to have a unique solution.
- As a consequence, two different solutions cannot start at the same time, t, at the same point in phase space.
- For autonomous systems, two different solutions that start at the same place in phase space but at different times will correspond to the same solution curve (because the direction field at each point will be the same, regardless of time).

Interpretation of the E & U Theorem

- This means that, for an autonomous system, solution curves cannot meet or cross in phase space.
- No such guarantee exists for solution curves of non-autonomous systems; solution curves for non-autonomous systems frequently cross in phase space.

Example 3: The phase portrait for the following differential equation is given below.

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -2.5 + y + x^2 + xy \end{aligned}$$



Important ideas from today's lecture:

- Numerical methods can be generalised to work for systems of DEs similarly to the way they work for single equations.
- 'Nice' IVPs have unique solutions.
- Solution curves for autonomous systems do not cross or meet in phase space.