Maths 260 Lecture 3

- **Topics for today:**
  Slope fields
  Euler's method
  Comparison of methods seen so far
- **Reading for this lecture:** BDH Section 1.3, 1.4
- **Suggested exercises:** BDH Section 1.3: 11, 13, 15; Section 1.4: 7
- **Reading for next lecture:** BDH Section 1.4 (again)
- **Today's handout:** "Lecture 3: Pictures from the lecture"

Slope Fields – a method for qualitative analysis of DEs

Slope fields help us visualise the graph of a solution to a DE without needing to first find a formula for the solution.

Assume $y(t)$ is a solution to

$$\frac{dy}{dt} = f(t, y).$$

Then at $t = t_1$, $y(t_1) = y_1$, and

$$\frac{dy}{dt} = f(t_1, y_1).$$

i.e., the slope of the solution at this point is $f(t_1, y_1)$.

We use this result to draw a **slope field**, which helps us sketch solutions to the DE.

To draw a slope field:

1. For selected points in the $t - y$ plane (say at all points on an evenly spaced grid) calculate $f(t, y)$.
2. For each point (t_1, y_1) selected in Step 1, draw a short line segment of slope $f(t_1, y_1)$ centred at the point (t_1, y_1).

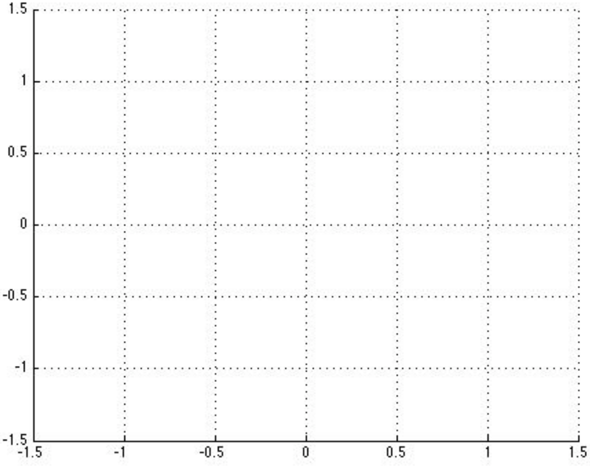
The resulting picture is called a slope field for the DE.

The Matlab function *dfield* plots slope fields.

Example 1:

Use the grid to draw the slope field for the DE

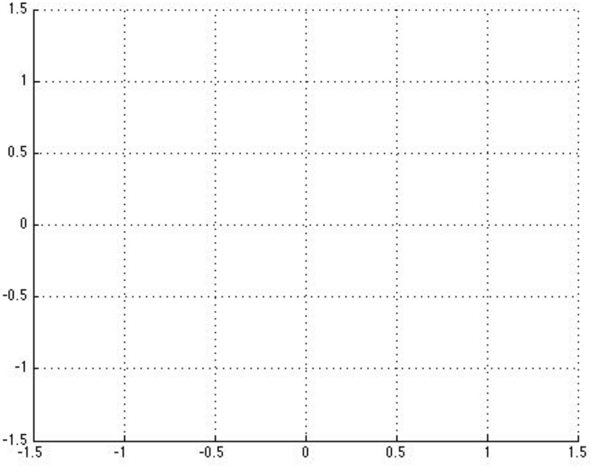
$$\frac{dy}{dt} = y - t$$



Example 2:

Draw the slope field for the DE

$$\frac{dy}{dt} = -yt$$



Sketching solutions using the slope field:

To sketch a solution to the IVP

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0$$

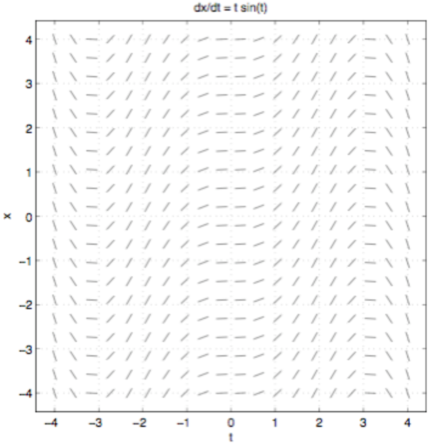
1. Sketch the slope field as above.
2. Starting at the point (t_0, y_0) draw a curve that follows the direction field.

Example 3:

The following picture shows the slope field for the DE

$$\frac{dy}{dt} = t \sin t$$

Sketch solutions to this DE satisfying: (a) $y(1) = 0$;
(b) $y(0) = -1$.

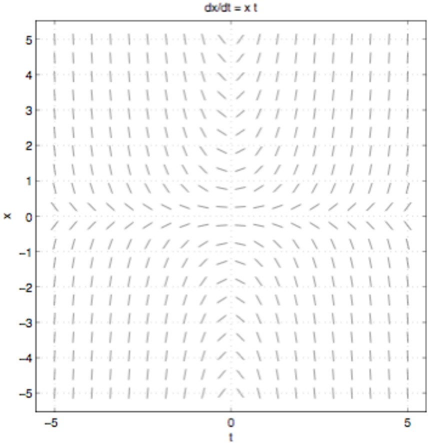


Example 4:

The following picture shows the slope field for the DE

$$\frac{dy}{dt} = yt$$

Draw solutions to this DE satisfying: (a) $y(1) = 0$; (b) $y(0) = -1$.



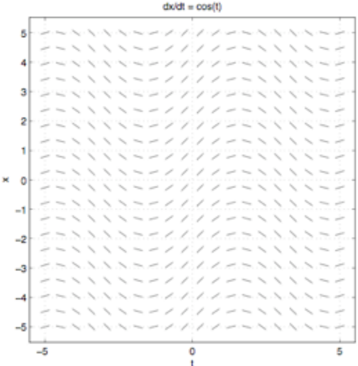
Two special cases:

For differential equations of the form

$$\frac{dy}{dt} = f(t)$$

the slope marks on each line of fixed t are parallel.

Example 5: $\frac{dy}{dt} = \cos t$



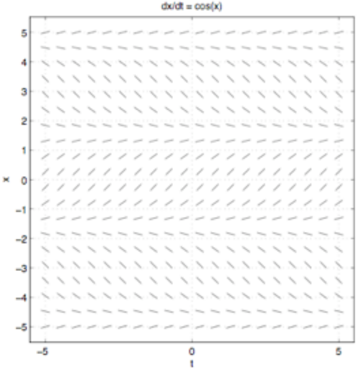
Graphs of different solutions are vertical translations of each other.

For differential equations of the form

$$\frac{dy}{dt} = f(y)$$

the slope marks on each line of fixed y are parallel.

Example 6: $\frac{dy}{dt} = \cos y$



The horizontal translation of a solution curve is also a solution.

Euler's method – a numerical method for analysis of DEs

We can obtain numbers and graphs that approximate solutions to IVPs using a class of techniques called numerical methods. Euler's method is the simplest numerical method.

Main idea of Euler's method:

For the IVP

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0$$

start at (t_0, y_0) and take small steps, with the direction of each step being the direction of the slope field at the start of that step.

More formally, given (t_0, y_0) and a stepsize, h, we want to calculate an approximation to $y(t_1), y(t_2), y(t_3)$, etc. where

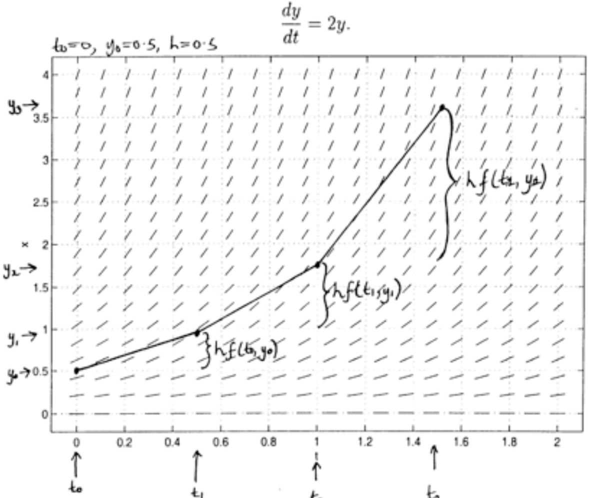
$$
\begin{aligned}
t_1 &= t_0 + h, \\
t_2 &= t_1 + h = t_0 + 2h, \text{ and so on.}
\end{aligned}
$$

Slope of field at (t_0, y_0) is $f(t_0, y_0)$ so

$$
\begin{aligned}
y(t_1) &\approx y_1 = y_0 + hf(t_0, y_0), \\
y(t_2) &\approx y_2 = y_1 + hf(t_1, y_1), \\
&\vdots \\
y(t_{k+1}) &\approx y_{k+1} = y_k + hf(t_k, y_k) \qquad \text{for } k = 0, 1, \ldots .n.
\end{aligned}
$$

The relationship between the slope field and the numerical solution obtained from Euler's method is shown below for the DE

$$\frac{dy}{dt} = 2y.$$



Example 7:

Use Euler's method to approximate the solution of

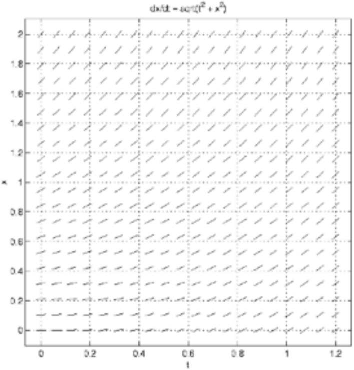
$$\frac{dy}{dt} = \sqrt{t^2 + y^2}, \qquad y(0) = 0.75$$

at $t = 0.25, t = 0.5, t = 0.75$, and $t = 1$

Solution:

n	t_n	y_n	$f(t_n, y_n)$	$y_n + hf(t_n, y_n)$
0	0.00	0.75	0.75	0.9375
1	0.25	0.9375	0.9703	1.1801
2	0.50	1.1801	1.2816	1.5005
3	0.75	1.5005	1.6775	1.9198
4	1.00	1.9198		

Compare this with the solution sketched using the slope field:



Comparison of methods seen so far

We have seen examples of three important types of methods for getting information about solutions to DEs.

Qualitative methods (e.g., slope fields) are useful for understanding qualitative properties of solutions (e.g., long term behaviour) but do not give exact values of solutions at particular times.

Analytic methods (e.g., solving separable equations) give a formula for a solution of a DE. These methods work in some important special cases but do not work in most cases.

Numerical methods (e.g., Euler's method) give approximate quantitative information about solutions. We can automate these methods (i.e., use a computer). These methods can be misleading, and give information about only one solution at a time.

It is usually possible to use more than one method for any problem - the trick is picking the most appropriate method(s).

We will learn more about each class of methods in the rest of the course.

Important ideas from today:

- How to sketch slope fields and use them to sketch solutions.
- Euler's method - a method for numerically approximating solutions to an IVP - is based on considering slope fields.
- Comparison of methods seen so far.