THE UNIVERSITY OF AUCKLAND

FIRST SEMESTER, 2005
Campus: City and Tamaki

MATHEMATICS

Differential Equations

(Time allowed: THREE hours)

NOTE: Answer all questions. Show all your working. 100 marks in total.

1. (15 marks) Find the general solution of the differential equations and solve the initial value problems. If the formula for your general solution does not give all solutions of the differential equation then find the missing solutions.

(a) $\dfrac{dy}{dt} = \dfrac{y}{t} - t^2, \quad y(1) = 0.$

(b) $\dfrac{dy}{dt} = y^2 \sin t, \quad y(\pi) = 1.$

2. (15 marks) Construct bifurcation diagrams for the following differential equations. Be sure to identify the value of μ at which bifurcation occurs.

(a) $\dfrac{dy}{dt} = y^2 - 2y + \mu.$

(b) $\dfrac{dy}{dt} = (y - \mu)(y - 2).$

3. (10 marks) Consider the initial value problem

$$\frac{dy}{dt} = \frac{t}{y}, \qquad y(1) = 2.$$

(a) Show that a solution to the initial value problem is $y(t) = \sqrt{t^2 + 3}$ (you do not need to solve the differential equation to answer this part of the question).

(b) Use the Improved Euler method to approximate the solution at $t = 3$, using step sizes of $h = 1$ and $h = 2$.

(c) Compute the error for each solution calculated in (b).

(d) Estimate (but do **not** calculate) the error you would obtain if you used the Improved Euler method to approximate the solution at $t = 3$, using step size of $h = 0.5$.

4. (15 marks) Consider the following two systems of differential equations:

(a) $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & 6 \\ -3 & -5 \end{pmatrix} \mathbf{Y}$ (b) $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \mathbf{Y}$

(i) Determine the general solution for each system. Express your answers in terms of real-valued functions.

(ii) Carefully sketch the phase portrait for each system.

(iii) Describe the long term behaviour of solutions in each system.

5. (20 marks) Consider the following system of equations:

$$\begin{aligned} \frac{dx}{dt} &= x + y + a \\ \frac{dy}{dt} &= y(x-1) \end{aligned}$$

where a is a real constant.

A grid is provided on the yellow answer sheet attached to the back of the question paper. Use the grid for your answer to part (b) of this question. Attach the yellow answer sheet to your answer book.

(a) Show that there are two equilibrium solutions and that one of these is the point $(-a, 0)$. What does linearisation tell you about the type (e.g. saddle, spiral, source) of equilibrium point at $(-a, 0)$?

(b) For $a = 0$ the equations become

$$\begin{aligned} \frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= y(x-1) \end{aligned}$$

(i) Determine the type (e.g., saddle, spiral source) of the other equilibrium solution (i.e., not the equilibrium solution at $(0, 0)$).

(ii) Find the nullclines for the system and sketch them on the answer sheet provided. Show the direction of the vector field in the regions between the nullclines and on the nullclines themselves.

(iii) Sketch the phase portrait for the system. Include in your phase portrait the solution curves passing through the initial conditions

A. $(x(0), y(0)) = (1, -2)$;

B. $(x(0), y(0)) = (-2, 2)$.

Make sure you show clearly where these solution curves go as $t \to \infty$.

6. (12 marks)

(a) Find the general solution to the following differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = e^{-3t}.$$

(b) Solve the following initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = \cos t, \quad y(0) = \frac{1}{5}, \ y'(0) = \frac{2}{5}.$$

7. (13 marks) In this question you may use the table of Laplace transforms attached.

(a) If

$$h(t) = \begin{cases} 0, & t < 1, \\ e^{-(t-1)}, & t \geq 1. \end{cases}$$

show that

$$\mathcal{L}\{h(t)\} = \frac{e^{-s}}{s+1}.$$

(b) Show that

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+1)^3}\right\} = \mathcal{U}_1(t)\frac{(t-1)^2}{2}e^{1-t}$$

where

$$\mathcal{U}_1(t) = \begin{cases} 0, & t < 1, \\ 1, & t \geq 1. \end{cases}$$

(c) Use the method of Laplace transforms and your answers to (a) and (b) to find a solution to the initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = h(t), \quad y(0) = 0, \ \left.\frac{dy}{dt}\right|_{t=0} = 1,$$

where $h(t)$ is as defined in (a).

A brief table of Laplace transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$t^n,\ n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$
$e^{at}t^n,\ n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$\mathcal{U}_a(t),\ a \geq 0$	$\frac{e^{-as}}{s}$
$f(t-a)\mathcal{U}_a(t),\ a \geq 0$	$e^{-as}F(s)$
$e^{at}f(t)$	$F(s-a)$
$\frac{d^n f}{dt^n}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{n-1}(0)$

ANSWER SHEET FOLLOWS

Candidate's Name: ______________________ ID No: ____________

TIE THIS ANSWER SHEET TO YOUR SCRIPT BOOK

Answer sheet for Question 5(b)