Maths 260 Lecture 26

- **Topic for today:**
  - More on classification of equilibria in nonlinear systems.
  - Using nullclines to sketch phase portraits for nonlinear systems.
- **Reading for this lecture:** BDH Section 5.2
- **Suggested exercises:** BDH Section 5.2; 1, 5, 7, 9, 11
- **Reading for next lecture:** BDH Section 5.2
- **Today's handouts:** None

Result from the last lecture

To determine the type of an equilibrium in a nonlinear system, we can sometimes use linearisation, i.e. we use a linear system to approximate the behaviour of solutions near the equilibrium in the nonlinear system.

For most systems, knowledge of the behaviour of solutions in the linearised system is sufficient to determine the behaviour near the corresponding equilibrium in the nonlinear system.

In particular, for the system

$$\frac{d\mathbf{Y}}{dt} = f(\mathbf{Y})$$

with an equilibrium $\mathbf{Y}(t) = \mathbf{Y_0}$, construct the linearised system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{J}(\mathbf{Y_0})\mathbf{Y}$$

where $\mathbf{J}(\mathbf{Y_0})$ is the Jacobian matrix of partial derivatives evaluated at $\mathbf{Y_0}$. If in the linearised system the equilibrium at the origin is a sink, source, or saddle, then $\mathbf{Y_0}$ is a sink, source, or saddle (respectively) in the nonlinear system.

Note that linearisation does not tell us anything about the behaviour of solutions to a nonlinear system far from an equilibrium.

Example 1

Find all equilibria and determine their types for the following system:

$$\frac{dx}{dt} = 2x$$
$$\frac{dy}{dt} = y(1-y)$$

For each equilibrium, sketch a phase portrait showing the behaviour of solutions near the equilibrium in the associated linear system.

$$J = \begin{pmatrix} 2 & 0 \\ 0 & 1 - 2y \end{pmatrix}$$

$$J(0,0) =$$

$$J(0,1) =$$

Unfortunately, linearisation does not always work.

In particular, if the Jacobian matrix has a **zero** eigenvalue or a **purely imaginary** eigenvalue, then we cannot predict the behaviour in the nonlinear system based on linearisation alone.

Example 2: Consider the system:

$$\frac{dx}{dt} = -x^3$$
$$\frac{dy}{dt} = -y + y^2$$

Equilibria:

Jacobian:

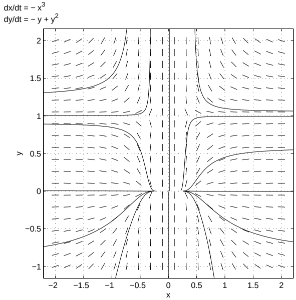
so at $(0, 0)$ the Jacobian is:

The linearised system has the phase portrait:

At $(0, 1)$, the Jacobian is:

The linearised system has the phase portrait:

However, the phase portrait for the nonlinear system is



- Notice that in this phase portrait, $(0,0)$ looks like a sink and $(0,1)$ looks like a saddle.
- These results were not predicted by the corresponding linearised systems.
- Linearisation does not work in these cases because of the zero eigenvalues of the Jacobians.

Sketching phase portraits for nonlinear systems

We would like to be able to sketch the complete phase portrait for a nonlinear system.

Linearisation gives us good information about the behaviour of solutions near most equilibria. We can use numerics to fill in the gaps — but it would be helpful to know in advance which regions of the phase space to look at numerically.

In order to do this we can use nullclines.

Definitions

Consider a system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

- The x-nullcline is the set of points (x, y) where $f(x, y) = 0$.
- The y-nullcline is the set of points (x, y) where $g(x, y) = 0$.

- On the x-nullcline, $\frac{dx}{dt} = 0$, and the vector field is vertical, pointing straight up or straight down.
- On the y-nullcline, $\frac{dy}{dt} = 0$, and the vector field is horizontal, pointing either left or right.
- At an intersection of an x-nullcline and a y-nullcline, $f(x, y) = g(x, y) = 0$, i.e. a point of intersection between an x-nullcline and a y-nullcline is an equilibrium solution.

Note that:

- A nullcline is not necessarily a solution curve.
- A nullcline is not necessarily a straight line.

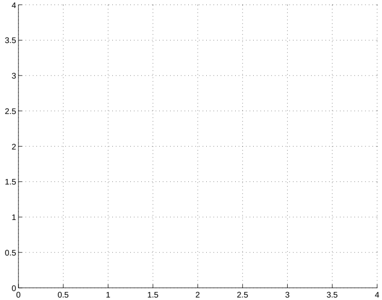
Example 3

Use nullclines to sketch the phase portrait for the system

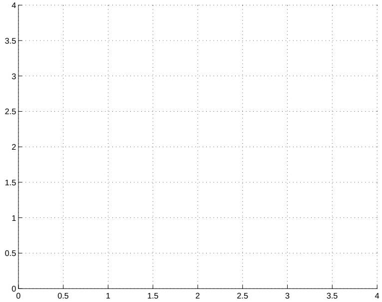
$$\frac{dx}{dt} = x(2 - x - y)$$
$$\frac{dy}{dt} = y(3 - 2y - x)$$

for $x, y \geq 0$.

The x-nullclines are:



The y-nullclines are:

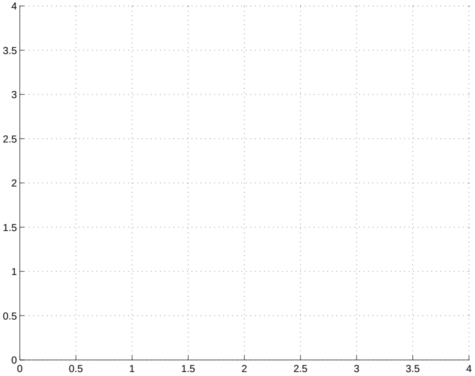


Nullclines divide the phase plane into regions where $\frac{dx}{dt}$ and $\frac{dy}{dt}$ have constant sign.

In the example above:

- $\frac{dx}{dt} > 0$ if
- $\frac{dx}{dt} < 0$ if
- $\frac{dy}{dt} > 0$ if
- $\frac{dy}{dt} < 0$ if

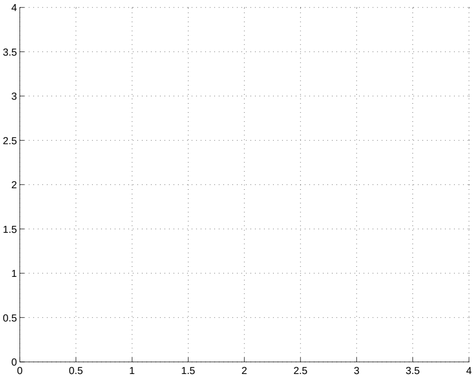
Combining information about x- and y-nullclines for this example, we get



Now the phase plane is divided into four regions:

- Region A: $\dot{x} < 0, \dot{y} < 0$
- Region B: $\dot{x} > 0, \dot{y} < 0$
- Region C: $\dot{x} > 0, \dot{y} > 0$
- Region D: $\dot{x} < 0, \dot{y} > 0$

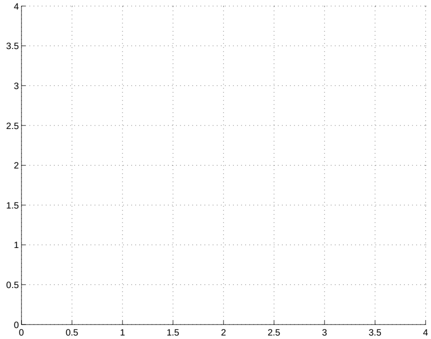
Use the known direction of solution curves in each region to determine the direction of solutions on the nullclines.



Now we can see that:

- Once solutions get into region B they cannot get out again. Solutions move down and right until they get to lower right corner, i.e. to the equilibrium at $(1, 1)$.
- Similarly, once solutions get into region D they cannot get out again. Solutions move up and left until they get to the upper left corner, i.e. to the equilibrium at $(1, 1)$.
- Solutions starting in region A or C must either leave the region by entering B or D (and then tend to $(1, 1)$) or must tend directly to $(1, 1)$.

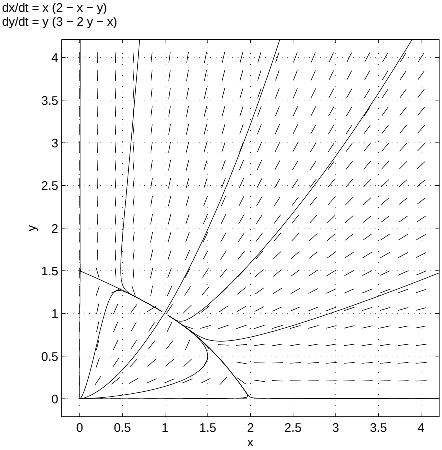
Hence, the phase portrait for the system must be, approximately:



This picture suggests that $(1,1)$ is a sink, $(0,0)$ is a source, and $(2,0)$ and $(0,3/2)$ are saddles.

Exercise: Use linearisation to confirm that this is so.

The approximate phase portrait obtained using nullclines looks very like the phase portrait obtained with pplane:



Important ideas from today:

- Definitions of sink, source and saddle in nonlinear systems.
- If the Jacobian matrix for a nonlinear system, evaluated at an equilibrium, has a zero or purely imaginary eigenvalue, then we cannot predict the behaviour near the equilibrium in the nonlinear system based on linearisation alone.
- Linearisation gives information about the behaviour of solutions near most equilibria, but is unhelpful far from those equilibria.
- Nullclines can be used to help sketch the complete phase portrait for a nonlinear system (including far from equilibria).