Test Solutions.

1. (10 marks)
Find a solution to the initial value problem

$$\frac{1}{t}\frac{dy}{dt} = \frac{2y}{t^2} + t\ \sin t,\ t > 0, \quad y(1) = 0.$$

Write as

$$\frac{dy}{dt} - \frac{2}{t}y = t^2 \sin t$$

Integrating factor is $e^{\int \frac{2}{t}dt} = t^{-2}$

$$\frac{d}{dt}\left(t^{-2}y\right) = \sin t$$

$$\Rightarrow \quad y = -t^2\cos t + c t^2$$

Now we find $y(1) = -\cos(1) + c = 0$

$\Rightarrow c = \cos(1)$

$$\Rightarrow \quad y(t) = -t^2(\cos t - \cos(1))$$

2. (8 marks)
Consider the following initial value problem

$$\frac{dy}{dt} = y + t,\ y(0) = 1.$$

(a) Does a unique solution of the IVP exist? Give reasons for your answer.

(b) Use Improved Euler with stepsize $h = 1$ to find an approximation to the solution at $t = 1$.

(a) In this case $f(y,t) = y + t$

so $f(y,t)$ is continuous and so is

$\frac{\partial f}{\partial y} = 1 \Rightarrow$ there exists a unique soln everywhere.

(b) Algorithm is

$y_1 = y_0 + \frac{h}{2}(m_1 + m_2)$ where

$m_1 = f(y_0, t_0) = f(1,0) = 1.$

& $m_2 = f(y_0 + h f(y_0, t_0), t_1)$

$= f(1 + 1, 1) = 3$

$\Rightarrow y_1 = 1 + \frac{1}{2}(1 + 3)$

$= 3$

3. (15 marks)
Consider the one-parameter family of differential equations

$$\frac{dy}{dt} = y(\mu - y^2),$$

where μ is the parameter.

(a) For $\mu = +1$, draw the phase line and use it to sketch a picture showing the behaviour of solutions in the $t - x$ plane. You do not need to find explicit formulas for the solutions you draw.

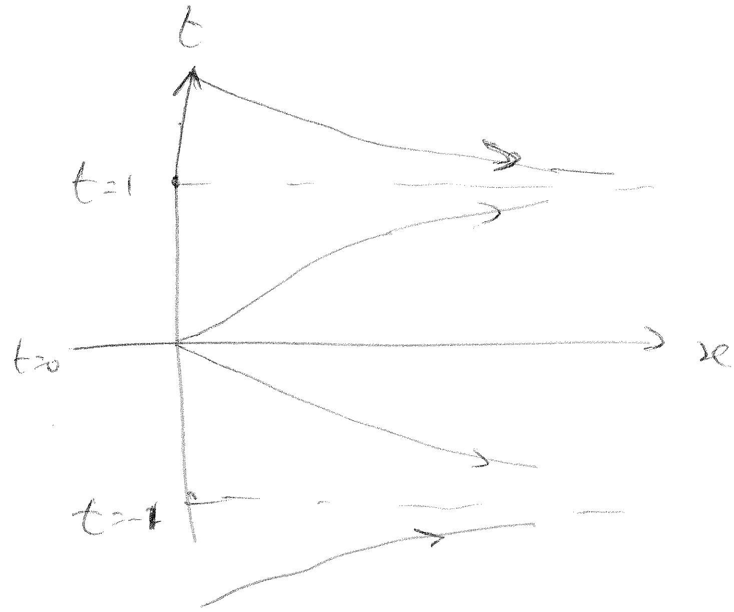
(b) For $\mu = -1$, draw the phase line. For this value of μ, describe the behaviour of solutions as t increases.

(c) Find all bifurcations and sketch the bifurcation diagram for the family of equations.

(a) $\mu = +1$ $\quad \frac{dy}{dt} = y(1-y^2)$

eq points at

$y=0$, $y=\pm 1$



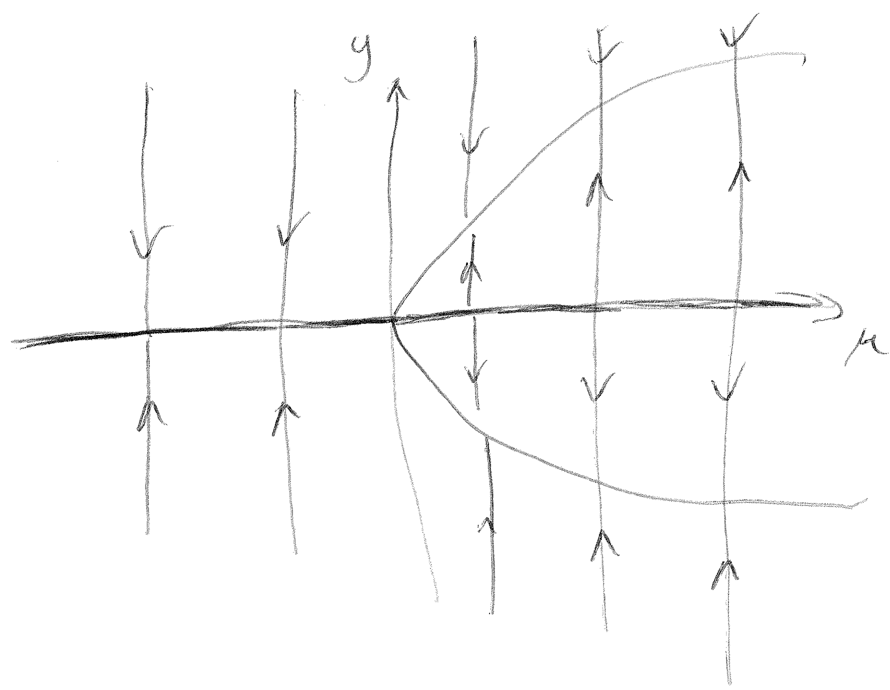
(blank page for your working)

(b) $\mu = -1$, $\dfrac{dy}{dt} = y(-1 - y^2)$

eq. pt $y = 0$

As t increase all solutions tend to the eq pt $y = 0$

(e)



The bifurcations occur at $\mu = 0$

4. (7 marks)

(a) The following is a model for the number P of fish in a lake,

$$\frac{dP}{dt} = kP - P^2 - P$$

where k is a positive constant. Show that if $k < 1$, then regardless of the initial number of fish, as $t \to \infty$ the number of fish will become zero.

(b) The Lotka-Voltera model for a predator-prey system is the following

$$\frac{dx}{dt} = ax - bxy$$

$$\frac{dy}{dt} = -my + nxy$$

where $x(0) = x_0$, $y(0) = y_0$ and $a, b, m, n > 0$. Identify which is the predator and which is the prey. What happens to the population of x ... as $t \to \infty$? if y becomes extinct

(a) If $k < 1$ then

$kP - P^2 - P < 0$ for all $P > 0$

therefore $\frac{dP}{dt} < 0$ so the population tends to zero.

(b)

y is the predator and x is the prey

If $y = 0$ then $\frac{dx}{dt} = ax$

$\Rightarrow x = Ae^{at}$ and the population grows to infinity as $t \to \infty$

5. (10 marks)
Consider the following system of differential equations

$$\frac{dx}{dt} = y$$

,

$$\frac{dy}{dt} = -y$$

.

(a) Find the general solution of the system of differential equations.

(b) Sketch a phase portrait showing the behaviour of typical solutions in the phase plane.

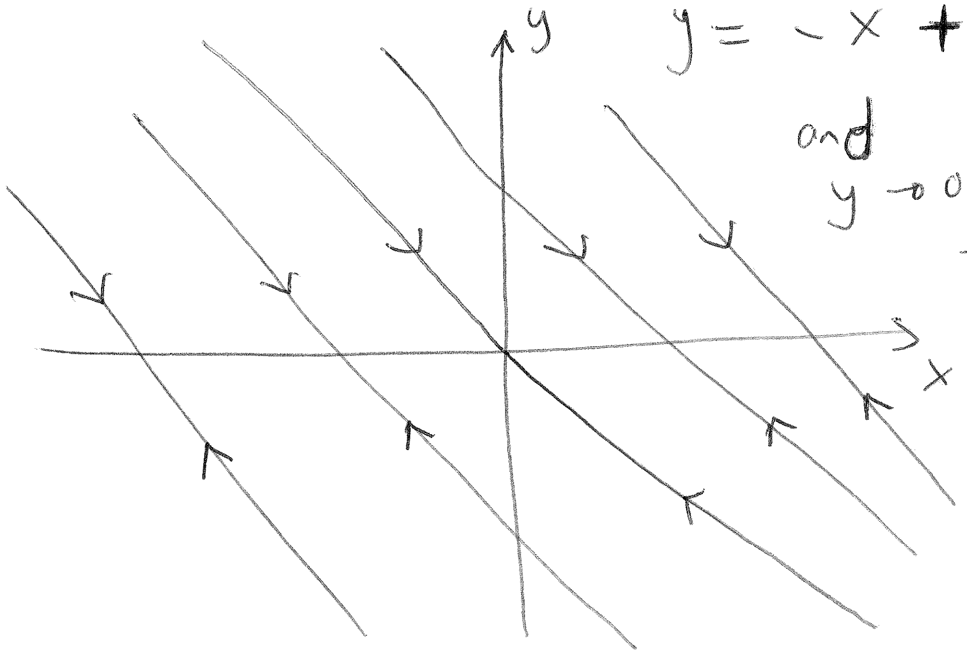
(a) First solve $\frac{dy}{dt} = -y$

$\Rightarrow y = c_1 e^{-t}$

then solve $\frac{dx}{dt} = y = c_1 e^{-t}$

$\Rightarrow x = -c_1 e^{-t} + c_2$

(b)



$y = -x + c_2$

and

$y \to 0$ as $t \to \infty$