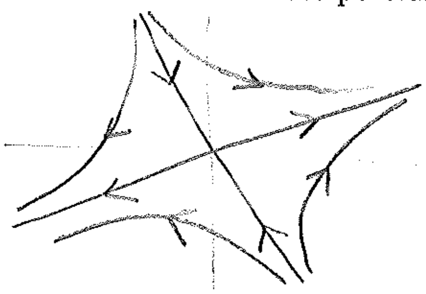
DEPARTMENT OF MATHEMATICS

Maths 260 Assignment 3 Solutions

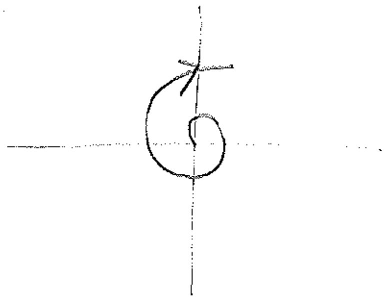
October 5, 2009 Due:

1. (22 marks)

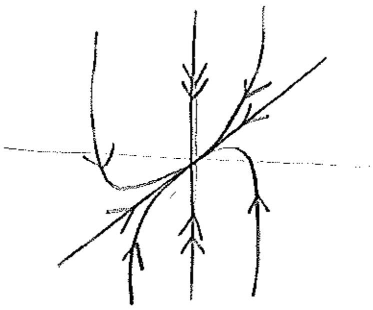
   (a) Eigenvalues satisfy $\lambda^2 - 3\lambda - 4 = 0$, so $\lambda = 4$, $\lambda = -1$, with eigenvectors $(4, 1)^T$, and $(1, -1)^T$ respectively, so zero is a saddle. Phase portrait:



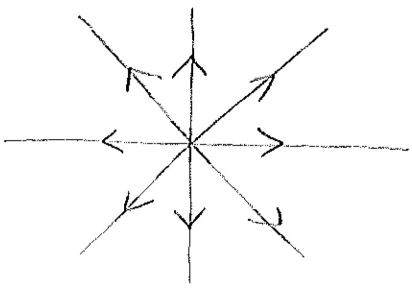
   (b) Eigenvalues satisfy $\lambda^2 - 2\lambda + 5 = 0$, so $\lambda = 1 \pm 2i$, so zero is a spiral source. To get the direction, we calculate $\mathbf{A}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\-4\end{pmatrix}$ so must be clockwise direction. Phase portrait:



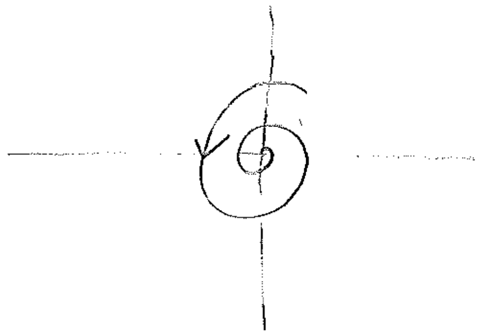
   (c) Eigenvalues are $\lambda = -1$, $\lambda = -2$, because matrix is triangular, with eigenvectors $(1, 1)^T$, and $(0, 1)^T$ respectively, so zero is a nodal sink. Phase portrait:



   (d) Eigenvalues are $\lambda = 1$ repeated, with two linearly independent eigenvectors $(0, 1)^T$, and $(1, 0)^T$, so zero is a star source. Phase portrait:



(e) Eigenvalues satisfy $\lambda^2 + 2\lambda + 15 = 0$, so $\lambda = -1 \pm i\sqrt{11}$, so zero is a spiral sink . To get the direction, we calculate $A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-3\\3\end{pmatrix}$ so must be anticlockwise direction. Phase portrait:



2. (7 marks)

(a) Euler's method is:

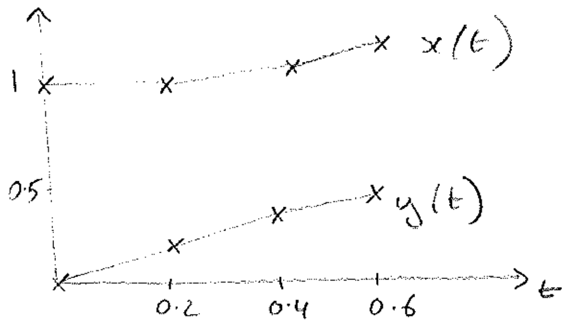
$$x_j = x_{j-1} + h\dot{x}(x_j, y_j, t_j)$$
$$y_j = y_{j-1} + h\dot{y}(x_j, y_j, t_j)$$

With $h = 0.2$ we get

j	t_j	x_j	y_j	$\dot{x}(x_j, y_j, t_j)$	$\dot{y}(x_j, y_j, t_j)$
0	0	1	0	0	1
1	0.2	1	0.2	0.4	0.8
2	0.4	1.08	0.36	0.7776	0.68
3	0.6	1.236	0.496		

giving $x(0.6) \approx 1.236$, $y(0.6) \approx 0.496$

(b) Sketch :



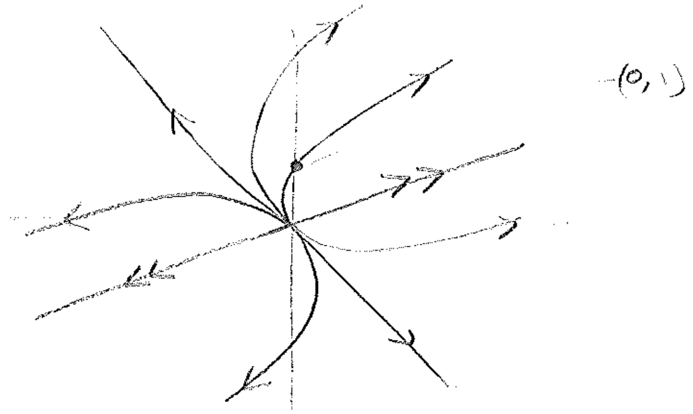
(c) Repeat with a smaller stepsize and/or use a higher order method such as RK4.

3. (6 marks)

(a) Eigenvalues satisfy $\lambda^2 - 7\lambda + 10 = 0$, so $\lambda = 5$, and $\lambda = 2$, with eigenvectors $(2,1)^T$, and $(1,-1)^T$ respectively. Hence the general solution is

$$\mathbf{Y} = c_1\begin{pmatrix}2\\1\end{pmatrix}e^{5t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{2t}$$

(b) Phase portrait:



4. (9 marks)

(a)

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & -1 & -2 \end{pmatrix} \mathbf{Y}$$

(b) Eigenvalues satisfy $(\lambda - 1)\left((-2-\lambda)(-2-\lambda)+1\right) = 0$, so $\lambda = 1$, and $\lambda = -2 \pm i$, with eigenvectors $(1,0,0)^T$ and $(0,1,\pm i)^T$ respectively.

We write

$$e^{(-2+i)t}\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} = e^{-2t}(\cos t + i \sin t)\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} = e^{-2t}\begin{pmatrix} 0 \\ \cos t \\ -\sin t \end{pmatrix} + ie^{-2t}\begin{pmatrix} 0 \\ \sin t \\ \cos t \end{pmatrix}$$

Hence the general solution is

$$\mathbf{Y} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t + c_2 e^{-2t}\begin{pmatrix} 0 \\ \cos t \\ -\sin t \end{pmatrix} + c_3 e^{-2t}\begin{pmatrix} 0 \\ \sin t \\ \cos t \end{pmatrix}$$

(c) Saddle.

(d) Solutions spiral inwards towards the vector $(1,0,0)^T$ and go out to infinity in this direction.

5. (7 marks)

(a)

$$\mathbf{A} \cdot \mathbf{v} = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1+i \end{pmatrix} = \begin{pmatrix} -1+2i \\ -3+i \end{pmatrix} = (-1+2i)\begin{pmatrix} 1 \\ \frac{-3+i}{-1+2i} \end{pmatrix}$$

and

$$\frac{-3+i}{-1+2i} = \frac{(-3+i)(-1-2i)}{(-1+2i)(-1-2i)} = \frac{5+5i}{5} = 1+i$$

So the eigenvalue is $-1+2i$.

(b)

$$\mathbf{Y} = c_1 \begin{pmatrix} 1 \\ 1+i \end{pmatrix} e^{(-1+2i)t}$$

(c)

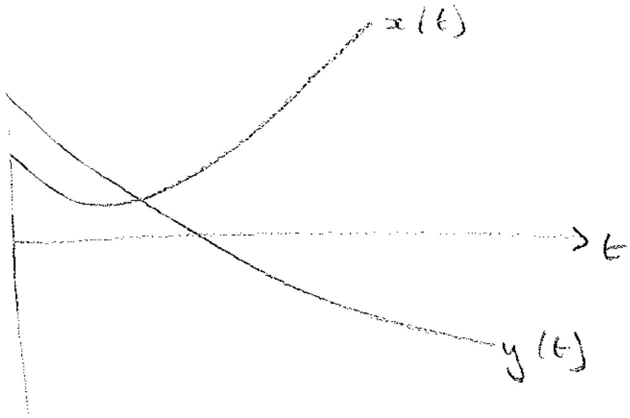
$$\begin{pmatrix} 1 \\ 1+i \end{pmatrix} e^{(-1+2i)t} = e^{-t}(\cos 2t + i\sin 2t)\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$
$$= e^{-t}\begin{pmatrix} \cos 2t + i\sin 2t \\ \cos 2t - \sin 2t + i\sin 2t + i\cos 2t \end{pmatrix}$$
$$= e^{-t}\begin{pmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix} + ie^{-t}\begin{pmatrix} \sin 2t \\ \sin 2t + \cos 2t \end{pmatrix}$$

Hence

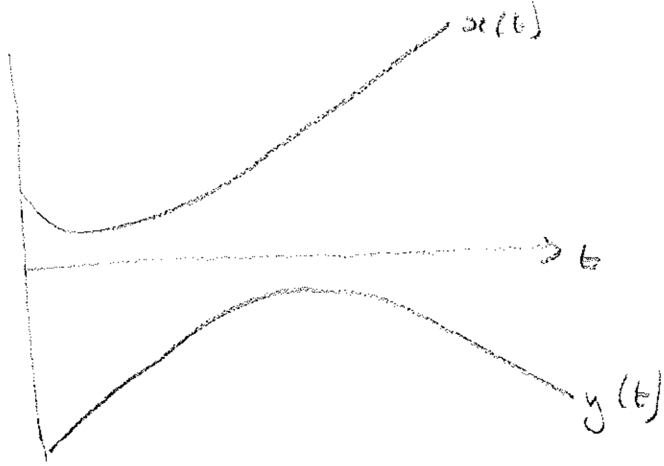
$$\mathbf{Y} = c_1 e^{-t}\begin{pmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin 2t \\ \sin 2t + \cos 2t \end{pmatrix}$$

6. (9 marks)

(a) i.



ii.



iii.

(b)