THE UNIVERSITY OF AUCKLAND

SECOND SEMESTER, 2007
Campus: City

MATHEMATICS

Differential Equations

(Time allowed: TWO hours)

NOTE: Answer **ALL** questions. Show **ALL** your working. 100 marks in total.

1. (15 marks)

(a) Solve the initial value problem

$$\frac{dy}{dt} = 2ty^2, \quad y(0) = 1.$$

(b) (i) Find the general solution of

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = t + 1.$$

Write your solution in terms of real-valued functions.

(ii) Describe the way solutions to this differential equation behave as t gets very large.

2. (15 marks) This question is about the differential equation

$$\frac{dy}{dt} = \frac{-2y}{t} + 4t, \quad t > 0.$$

(a) Show that

$$y(t) = t^2 + \frac{c}{t^2}$$

is a solution to the differential equation for $t > 0$, where c is an arbitrary constant.

(b) Now consider the initial value problem consisting of the differential equation given above together with the initial condition $y(1) = 2$. Use one step of the Improved Euler method to calculate an approximate value of solution of the initial value problem at final time $t = 2$.

(c) Compute the error in the approximation to $y(2)$ that you calculated in (b). Hint: use the solution given in (a) with an appropriate choice of the constant c.

(d) A better approximation to $y(2)$ is required. State two ways you could modify your calculations in (b) to obtain an approximation with better accuracy. You do not need to do any further calculations to answer this part of the question.

(e) Find all values of t_0 and y_0 such that there is a unique solution to the initial value problem consisting of the differential equation above together with the initial condition $y(t_0) = y_0$.

3. (16 marks) Construct a bifurcation diagram for the differential equation

$$\frac{dy}{dt} = y(1 - y) + \mu.$$

Identify any value of μ at which there is a bifurcation. Be sure to label the main features of your bifurcation diagram. Show all your working.

4. (14 marks) Consider the following system of differential equations:

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -4 & a \\ -3 & 0 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

(a) Find all values of a for which the equilibrium solution at $(x, y) = (0, 0)$ is a saddle.

(b) Find all values of a for which the equilibrium solution at $(x, y) = (0, 0)$ is a spiral sink.

(c) For the choice $a = 1$, find the general solution to the differential equation. Sketch the corresponding phase portrait, showing various solutions including the solution satisfying $(x(0), y(0)) = (-1, 0)$.

5. (25 marks) Consider the following system of equations:

$$\frac{dx}{dt} = y(1-x)$$
$$\frac{dy}{dt} = x(1-y)$$

A grid is provided on the yellow answer sheet attached to the back of the question paper. Use the grid for your answer to parts (b) and (c) of this question. Attach the yellow answer sheet to your answer book.

(a) Find the two equilibrium solutions and determine their type (e.g., spiral source, saddle).

(b) Find the nullclines for the system and sketch them on the answer sheet provided. Show the direction of the vector field in the regions between the nullclines and on the nullclines themselves.

(c) Sketch the phase portrait of the system.Your phase portrait should show the behaviour of solutions near the equilibria, and should show various solution curves *including* those passing through the following initial conditions:

(i) $(x(0), y(0)) = (-2, 0)$;
(ii) $(x(0), y(0)) = (2, 1)$;
(iii) $(x(0), y(0)) = (0, -1)$.

Make sure you show clearly where solution curves go as $t \to \infty$.

6. (15 marks)
Tall buildings such as the Sky Tower are designed to sway in the wind. This makes them less expensive to build. However, the designers of such buildings need to be able to predict how the swaying will occur in order to make sure the building will not fall over. A possible model for the swaying of the Sky Tower is:

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx + cx^3 = 0$$

where $x(t)$ is the displacement of the top of the building from the vertical axis caused by a gust of wind, measured in metres.

(a) Briefly describe (i.e., in one or two paragraphs) the methods you could use to get information about solutions to this model of the Sky Tower. You do not need to do any calculations to answer this part of the question.

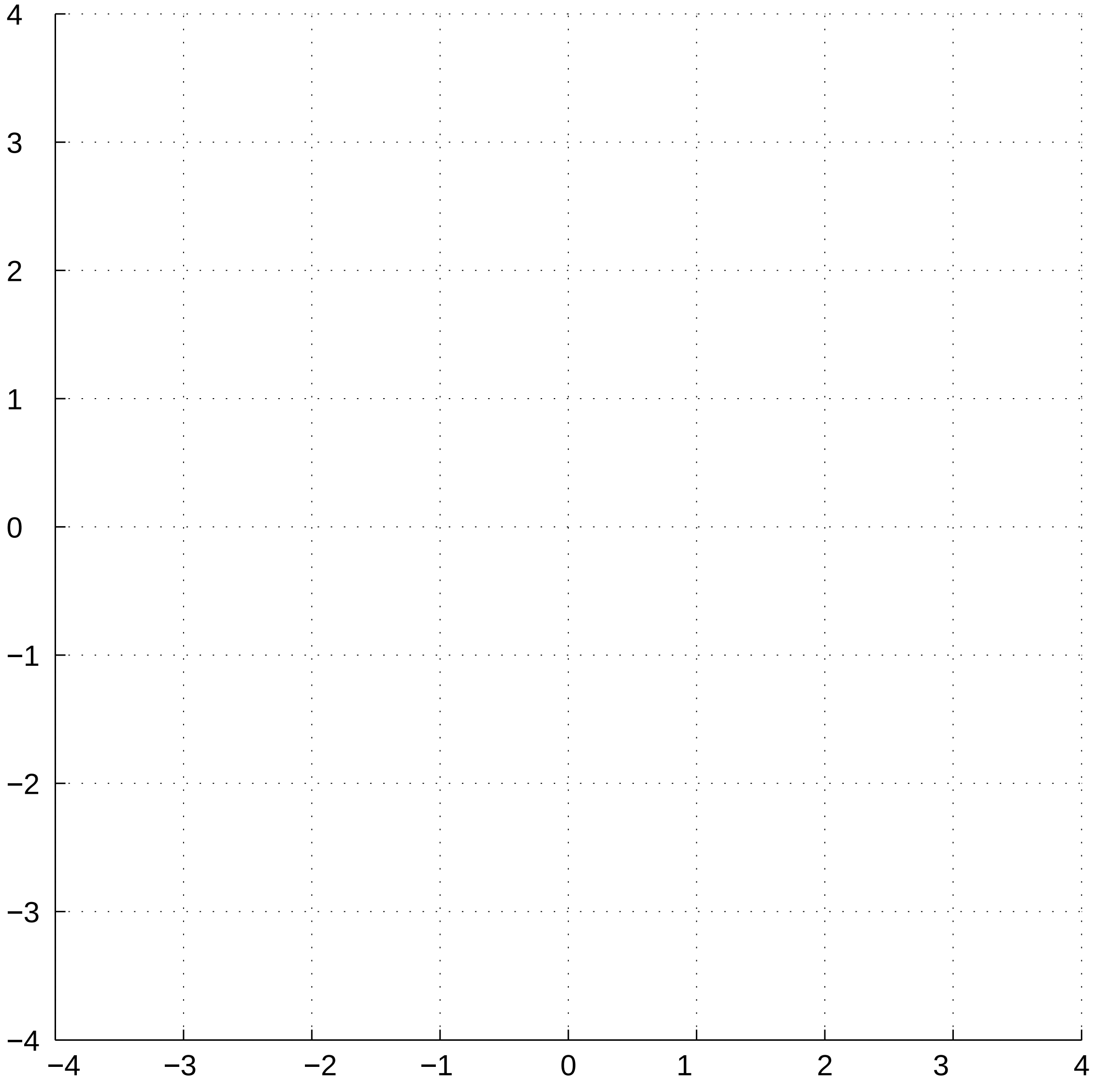
(b) Briefly describe what each of the first three terms in the differential equation might represent (i.e., say why these terms are included in the model). Say whether there will be any conditions on the signs or sizes of the constants a and b.

(c) What do you think the fourth term in the differential equation, cx^3, might represent?

Candidate's Name: ______________________ ID No: ____________

TIE THIS ANSWER SHEET TO YOUR ANSWER BOOK

Answer sheet for Question 5



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