Maths 260 Lecture 18

- **Topics for today:**
  Classification of equilibria in linear systems with real eigenvalues
- **Reading for this lecture:** BDH Section 3.3
- **Suggested exercises:**
  BDH Section 3.3 #1,5,9,11,19
- **Reading for next lecture:** BDH Appendix C
- **Today's handouts:** none

Linear Systems

- This lecture looks at systems of the form

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

  where **A** is a matrix with real eigenvalues.

- All such systems have an equilibrium at the origin. We are interested in the behaviour of solutions near the origin, especially when viewed in phase space.

Example 1

Determine the behaviour of solutions to the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \mathbf{Y}.$$

Sketch the phase portrait.

- The general solution is

$$\mathbf{Y}(t) = c_1 e^{3t} \begin{pmatrix} 6 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

- To see the behaviour of solutions that are not straight-line solutions, i.e., solutions with $c_1 \neq 0$ and $c_2 \neq 0$, note that as $t \to \infty$

$$\mathbf{Y}(t) = c_1 e^{3t} \begin{pmatrix} 6 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \to c_1 e^{3t} \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

i.e., as $t \to \infty$, these solutions behave like the straight-line solution

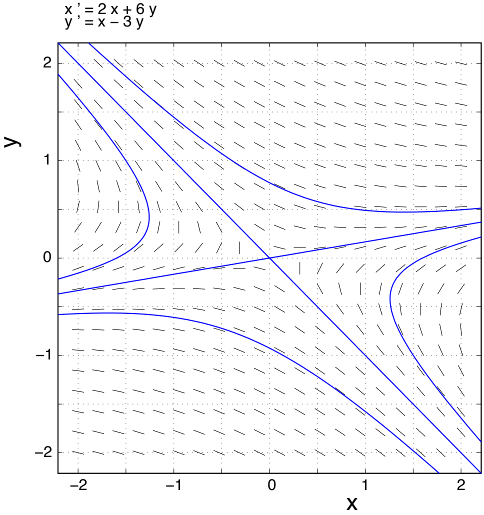
$$c_1 e^{3t} \begin{pmatrix} 6 \\ 1 \end{pmatrix}.$$

- Similarly, as $t \to -\infty$, these solutions behave like the straight-line solution

$$c_2 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

- We use these observations to sketch the phase portrait:

- Direction field and some solutions from *pplane*



- Note that on solution curves for the straight-line solution

$$\mathbf{Y}_1(t) = c_1 e^{3t} \begin{pmatrix} 6 \\ 1 \end{pmatrix},$$

  the arrows point away from the origin, because
  $\mathbf{Y}_1(t) \to \mathbf{0}$ as $t \to -\infty$.

- Similarly, arrows on the solution curves for the straight-line solution

$$\mathbf{Y}_2(t) = c_2 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

  point towards the origin, because $\mathbf{Y}_2(t) \to \mathbf{0}$ as $t \to \infty$.

Saddles

- This example illustrates typical behaviour of solutions to a planar linear system with *one positive real eigenvalue* and *one negative real eigenvalue*.
- A characteristic feature of phase portrait is the presence of two special lines:
  - On one line, solutions tend to origin as $t \to -\infty$.
  - On the other line, solutions tend to the origin as $t \to \infty$.
  - All other solutions tend to ∞ as $t \to \pm\infty$.
- The equilibrium point at the origin in this type of system is called a **saddle**.

Example 2:

Determine the behaviour of solutions to the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -4 & -2 \\ -1 & -3 \end{pmatrix} \mathbf{Y}.$$

- The general solution is

$$\mathbf{Y}(t) = c_1 e^{-5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

- Note that

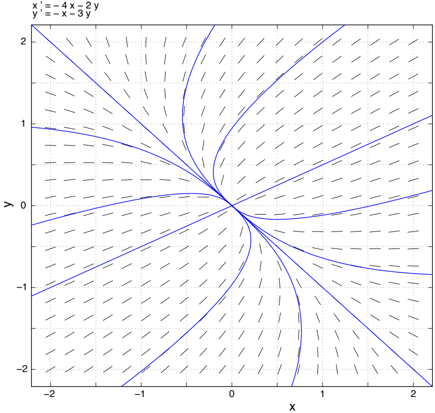
$$e^{-5t} \to 0 \text{ and } e^{-2t} \to 0, \text{ as } \ t \to \infty,$$

  and so all solutions tend to the origin as t increases.
- This is a general result: if all eigenvalues of matrix **A** are real and negative, then all solutions to the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

  tend to the origin as t increases.

- Direction field and some solutions:



- This picture suggests that most solutions are tangent to the straight-line solution

$$e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

- We can prove this is the case, as follows.
- The slope of a solution curve is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

- so, if $c_2 \neq 0$,

$$\lim_{t \to \infty} \left(\frac{dy}{dx} \right) = -1.$$

- Thus as $t \to \infty$, all solutions tend to the origin and almost all are tangent to the straight-line solution

$$e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Sinks

- In general, in a linear system with *two real, negative eigenvalues* λ_1 and λ_2, with $\lambda_1 < \lambda_2 < 0$, all solutions tend to the origin as $t \to \infty$.
- The equilibrium point in this type of system is called a **sink.**
- Except for those solutions starting on the line of eigenvectors corresponding to λ_1, all solutions are tangent at $(0, 0)$ to the line of eigenvectors corresponding to λ_2.
- This means that almost all solutions are tangent to the eigenvector corresponding to the slow eigenvalue, i.e., the eigenvalue closest to zero.

Example 3

Determine the behaviour of solutions to the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{Y}.$$

- The general solution is

$$\mathbf{Y}(t) = c_1 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

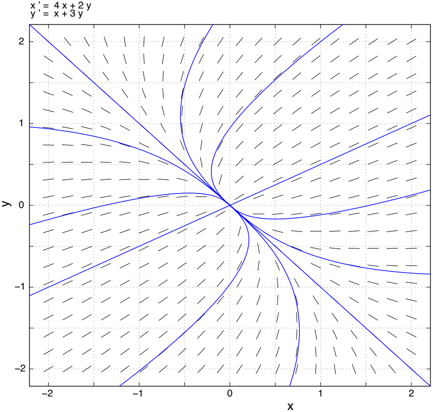
- As $t \to \infty$, all non-zero solutions move away from the origin.
- This is a general result: if all eigenvalues of $\mathbf{A}$ are real and positive, all non-zero solutions to the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

  tend away from the origin as $t \to \infty$.

- The equilibrium point in this case is called a **source.**

- Direction field and some solutions:



- This picture suggests that as $t \to -\infty$ most solutions are tangent to the straight-line solution

$$e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

- We can prove this either:
  - by the method used in Example 2, or
  - by noting that this example corresponds to reversing time in Example 2. Hence, the phase portrait is the same as in last example but with direction of arrows reversed.
- This is a general result, i.e., if **A** is a 2×2 matrix with eigenvalues λ_1 and λ_2 , with $0 < \lambda_2 < \lambda_1$, then except for those solutions starting on the line of the eigenvectors corresponding to λ_1 all solutions are tangent at $(0, 0)$ to the line of eigenvectors corresponding to λ_2.
- This means that almost all solutions are tangent to the eigenvector corresponding to the 'slow' eigenvalue, i.e., the eigenvalue closest to zero. This is just as we found for the behaviour of solutions near a sink.

Higher Dimensions

- This classification of equilibria extends to higher dimensions:
- For the system

  $$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

  $\mathbf{Y}(t) = 0$ is always an equilibrium.
- Assuming that all eigenvalues of **A** are real and distinct, then:
  - If all eigenvalues of **A** are positive, $\mathbf{Y}(t) = 0$ is a **source.**
  - If all eigenvalues of **A** are negative, $\mathbf{Y}(t) = 0$ is a **sink.**
  - If at least one eigenvalue of **A** is negative and at least one eigenvalue is positive, $\mathbf{Y}(t) = 0$ is a **saddle.**

Example 4

Draw the phase portrait for the system of equations

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \mathbf{Y}.$$

Example 5

Draw the phase portrait for the system of equations

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} \mathbf{Y}.$$

Example 6

Draw the phase portrait for the system of equations

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & 3 \\ 0 & -4 \end{pmatrix} \mathbf{Y}.$$

Important ideas from today's lecture:

For the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

in the case that all eigenvalues of $\mathbf{A}$ are real and distinct:

- If all the eigenvalues are positive, $\mathbf{Y}(t) = 0$ is a **source.**
  All solutions tend to the origin as $t \to -\infty$.
- If all the eigenvalues are negative, $\mathbf{Y}(t) = 0$ is a **sink.**
  All solutions tend to the origin as $t \to \infty$.
- If at least one eigenvalue is negative and at least one eigenvalue is positive, $\mathbf{Y}(t) = 0$ is a **saddle.**
  Most solutions tend to ∞ as $t \to \pm\infty$.

In the case of a sink or a source, solution curves are tangent at the origin to the "slow" eigenvector, i.e., the eigenvector with corresponding eigenvalue closest to zero.