Department of Mathematics
Maths 260 Differential Equations

Some notes on complex numbers

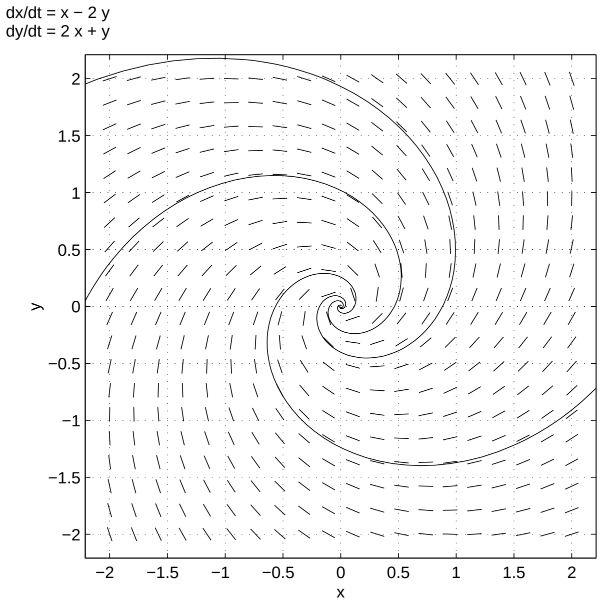
1 Introduction

There exist linear systems for which there are no straight-line solutions.

Example: Consider the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \mathbf{Y}.$$

Slope field and some solutions



What goes wrong? Why are there no straightline solutions?

Calculate the eigenvalues:

$$0 = \det \begin{pmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{pmatrix} = \lambda^2 - 2\lambda + 5.$$

So the quadratic formula gives:

$$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm \sqrt{-16}}{2}.$$

We need the square root of a negative number!

Let's suppose that we know what the square root of -1 is. We'll call it i. Then we could simplify our expression for λ:

$$\lambda = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4\sqrt{-1}}{2} = 1 \pm 2i.$$

This is an example of a complex number. Since complex numbers show up in the theory of differential equations (and in lots of other areas of mathematics), we need a good understanding of them. These notes give a basic introduction to complex numbers. Another good source of information about complex numbers for use in Maths 260 is Appendix C of the course textbook, "Differential Equations (3rd edition)", by Blanchard, Devaney and Hall.

2 Complex Numbers

Definition: An expression $a+bi$, where a and b are real numbers, is called a complex number. a is called the real part of the complex number, b is called the imaginary part.

Notation: $a = \text{Re } z, b = \text{Im } z$.

We can define the following operations on complex numbers:

1. **Addition:**

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

   Example: $(5 + 3i) + (6 - 7i) = 11 - 4i$.

2. **Subtraction:**

$$(a + bi) - (c + di) = (a - c) + (b - d)i.$$

3. **Multiplication:**

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

4. **Division:**

$$(a + bi)/(c + di) = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i.$$

Note: The normal rules of algebra apply to complex numbers, i.e., complex numbers obey distributive, associative, commutative rules etc., just like real numbers. Because of this, we do not need to memorise the definitions of the operations. Just use the fact $i^2 = -1$ and apply the usual rules of algebra.
Example:

$$\begin{aligned}(5 + 4i)(6 - 7i) &= 30 + 24i - 35i - 28i^2 \\ &= 30 + 24i - 35i + 28 \\ &= 58 - 11i\end{aligned}$$

Definition: $a - bi$ is called the complex conjugate of $a + bi$.

Notation: If $z = a + bi$, then $\bar{z} = a - bi$ denotes the complex conjugate of z.

Notice that

$$z\bar{z} = (a + bi)(a - bi) = a^2 + abi - abi + b^2 = a^2 + b^2$$

is a real quantity.

Division rule: When doing division by a complex number, e.g., working with an expression of the form

$$\frac{a + bi}{c + di},$$

it is often helpful to multiply the numerator and denominator by the conjugate of the numerator, as illustrated in the following example.

Example:

$$\frac{5 + 2i}{3 - 4i} = \left(\frac{5 + 2i}{3 - 4i}\right)\left(\frac{3 + 4i}{3 + 4i}\right) = \frac{15 - 8 + 6i + 20i}{3^2 + 4^2} = \frac{7 + 26i}{25}$$

Since $i^2 = -1$, we can think of i as being a square root of -1 ($i = \sqrt{-1}$). Another square root of -1 is $-i$. This is another way of saying that i and $-i$ are the roots of the polynomial $m^2 + 1 = 0$. It is no coincidence that the two roots of this polynomial are complex conjugates of one another, as complex roots of a polynomial always occur in complex conjugate pairs if the polynomial has real coefficients.

Example: The polynomial

$$m^2 + 2m + 2 = 0$$

has solutions

$$\begin{aligned} m &= = \frac{-2 \pm \sqrt{-4}}{2} \\ &= \frac{-2 \pm 2\sqrt{-1}}{2} = -1 \pm i. \end{aligned}$$

So $m_1 = -1 - i$ and $m_2 = -1 + i$ are two solutions, and $m_2 = \bar{m}_1$ as expected.

3 Complex Plane (or Argand diagram)

Since complex numbers are determined by two real numbers, it is natural to plot them on the usual coordinate plane. The vertical axis is called the *imaginary axis* and the horizontal axis is called the *real axis*. The real axis consists of purely real numbers. The imaginary axis consists of points of the form bi; these are called purely imaginary numbers. Complex conjugates are mirror images of each other in the real axis.

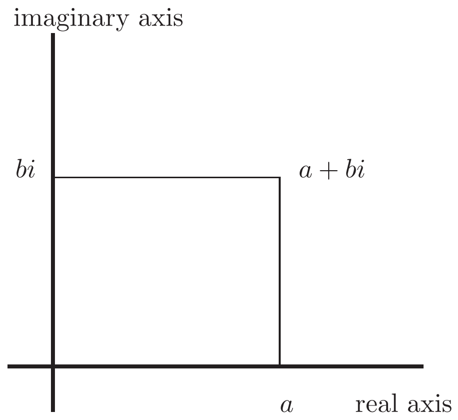


Figure 1: Argand Diagram

Definition: The absolute value (or modulus) of a complex number $z = a + bi$ is defined to be $|z| = \sqrt{a^2 + b^2}$. Notice that $|z|$ is the distance between the origin and the point z.

Example: $z = 3 - 4i$, $|z| = \sqrt{9 + 16} = 5$.

Notice that $|z| = \sqrt{z}$ or $|z|^2 = z\bar{z}$ because $z\bar{z} = a^2 + b^2$.

4 The polar form of complex numbers

The complex number $a + bi$ can be written in the polar form

$$z = r(\cos\theta + i\sin\theta) \text{ (polar form)}$$

if r and θ are defined so that $a = r\cos\theta$ and $b = r\sin\theta$, where $r = |z|$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$. θ is called an *argument* of z and is denoted arg z. See Figure 2.

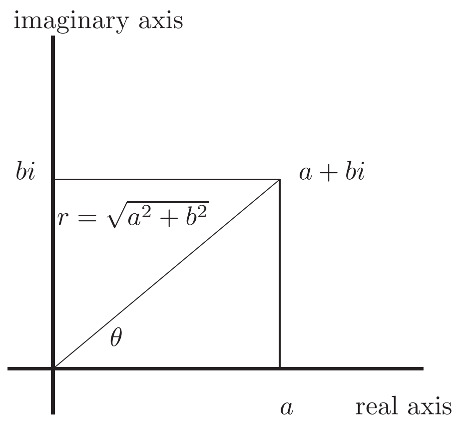


Figure 2: Polar coordinates

Notice that if θ is an argument of z then so is $\theta + 2m\pi$, for any integer m. Sometimes it is useful to restrict arguments to be in $[0, 2\pi)$ or some other interval of length 2π. Such arguments are called *principal arguments* and are often denoted Arg z.

Multiplication of polar forms: Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

be any two complex numbers, then

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)) \\ &= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \end{aligned}$$

Therefore, the product of two complex numbers is the complex number whose magnitude is the product of the magnitudes of the two factors, and whose argument is the sum of the arguments of the factors.

This in turn gives us:

De Moivre's formula

$$(\cos(\theta) + i\sin(\theta)^n = \cos(n\theta) + i\sin(n\theta),$$

a very useful formula...

Example: Express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta, \sin\theta$.

From de Moivre's formula, we have

$$\begin{aligned} \cos(3\theta) + i\sin(3\theta) &= (\cos(\theta) + i\sin(\theta))^3 \\ &= \cos^3(\theta) + i3\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta)) \\ &= \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) + i(3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)). \end{aligned}$$

So

$$\begin{aligned} \cos(3\theta) &= \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) \\ \sin(3\theta) &= 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta). \end{aligned}$$

Polar forms are sometimes useful for solving equations.

Example: Solve $z^3 = 1$.
$z = 1$ is obviously a solution. Any others? Let's write

$$z = r(\cos\theta + i\sin\theta),$$

where $r = |z| > 0$. Then

$$z^3 = r^3(\cos 3\theta + i\sin 3\theta)$$

and therefore

$$r^3(\cos 3\theta + i\sin 3\theta) = 1$$

and so

$$r^3 = 1, \text{ and } 3\theta = 0 + 2n\pi \Leftrightarrow \theta = \frac{2n\pi}{3}, n = 0, 1, 2...$$

So the solutions will be

$$z = \cos 0 + i \sin 0,\ \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$

i.e.,

$$z = 1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Notice that for $n = 3, 4...$, the solutions given coincide with the above solutions because of the periodicity of cosine and sine.

Note: The Fundamental Theorem of Algebra says that a polynomial equation of degree n has n roots (some of these may be repeated). More precisely,

$$a_n z^n + a_{n-1} z^{n-1} ... + a_0 = a_n(z - z_1)(z - z_2)...(z - z_n).$$

5 Derivatives of a complex-valued function of a real variable

$$f(t) = u(t) + iv(t).$$

We define

$$f'(t) = u'(t) + iv'(t).$$

Example: If $f(t) = 3t + i\sin t$ then $u(t) = 3t$ and $v(t) = \sin t$ so $f'(t) = 3 + i\cos t$.

6 Complex exponentials

Consider the complex-valued function

$$f(\theta) = \cos(\theta) + i\sin(\theta).$$

This function has the properties

- $f'(t) = if(t)$
- $f(0) = 1$
- $f(t_1)f(t_2) = f(t_1 + t_2)$

These are similar properties to the function $g(t) = e^{at}$, where a is real:

- $g'(t) = ag(t)$
- $g(0) = 1$
- $g(t_1)g(t_2) = g(t_1 + t_2)$

The similarities between the properties of f and g prompted Euler to make the definition:

Definition: If θ is real,

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{(Euler's formula)}$$

Euler discovered this when he was working on differential equations!

Exercise: Rewrite the following complex numbers in the form $a+bi$: $e^{i\pi}$, $e^{i\pi/2}$, $e^{i\pi/4}$.

The exponential of other complex numbers: If $z = x + iy$, we define

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y).$$

Note that

$$\begin{aligned} e^{z_1}e^{z_2} &= e^{x_1+iy_1}e^{x_2+iy_2} \\ &= e^{x_1}e^{x_2}e^{iy_1}e^{iy_2} \\ &= e^{x_1+x_2}e^{iy_1+iy_2} \\ &= e^{z_1+z_2} \end{aligned}$$

Theorem: If m is a complex number, then

$$\frac{d}{dt}e^{mt} = me^{mt}.$$

Proof: Write $m = a + ib$. Then

$$\begin{aligned} e^{mt} = e^{at+ibt} & \\ &= e^{at}(\cos(bt) + i\sin(bt)) \\ &= u(t) + iv(t). \end{aligned}$$

where $u(t) = e^{at}\cos(bt), v(t) = e^{at}\sin(bt)$. Clearly,

$$\begin{aligned} \frac{du}{dt} &= ae^{at}\cos(bt) - be^{at}\sin(bt) \\ \frac{dv}{dt} &= ae^{at}\cos(bt) + be^{at}\sin(bt). \end{aligned}$$

Then

$$\frac{d}{dt}e^{mt} = ae^{at}\cos(bt) - be^{at}\sin(bt) + iae^{at}\cos(bt) + ibe^{at}\sin(bt).$$

But

$$\begin{aligned} me^{mt} &= (a+ib)e^{at}(\cos(bt) + i\sin(bt)) \\ &= ae^{at}\cos(bt) - be^{at}\sin(bt) + iae^{at}\cos(bt) + ibe^{at}\sin(bt). \end{aligned}$$

Thus, if m is a complex number

$$\frac{d}{dt}e^{mt} = me^{mt}.$$

Example:

$$\frac{d}{dt}e^{(3+i)t} = (3+i)e^{(3+i)t}.$$

7 Complex eigenvalues and eigenvectors

In Maths 260, we frequently need to calculate complex eigenvalues and eigenvectors. The procedure is the same as for real eigenvalues and eigenvectors, but the calculations can *look* trickier because of the complex algebra.

Example: Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

The eigenvalues are the roots of

$$\det \begin{pmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 4 = \lambda^2 - 2\lambda + 5 = 0,$$

i.e., $\lambda = 1 \pm 2i$. To find the eigenvector corresponding to $\lambda = 1+2i$, we must find $\mathbf{v_1}$ such that

$$\begin{pmatrix} 1-(1+2i) & -2 \\ 2 & 1-(1+2i) \end{pmatrix} \mathbf{v_1} = 0.$$

Writing

$$\mathbf{v_1} = \begin{pmatrix} x \\ y \end{pmatrix},$$

we see that $-2ix - 2y = 0$ or $y = -ix$. Hence,

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

is an eigenvector, as is any constant multiple of this.

A similar calculation shows that

$$\mathbf{v_2} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

is an eigenvector corresponding to the eigenvalue $\lambda = 1 - 2i$.

Exercise: Show that the eigenvalues of the matrix

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 0 & 4 & -4 \\ 0 & 1 & 4 \end{pmatrix}$$

are $4, 4+2i, 4-2i$ with eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1-i/2 \\ 2i \\ 1 \end{pmatrix}, \begin{pmatrix} 1+i/2 \\ -2i \\ 1 \end{pmatrix},$$

respectively.

Note: The last two examples illustrate the following points: if a matrix has only real entries, then any complex eigenvalues will come in complex conjugate pairs; also, if $\mathbf{v}$ is the eigenvector corresponding to a complex eigenvalue λ, then $\bar{v}$ is the eigenvector corresponding to the eigenvalue $\bar{\lambda}$, i.e., the eigenvectors come in complex conjugate pairs too.

It is not always obvious from inspection when a vector with complex entries is a constant multiple of another vector, but it is easy to check by multiplication.

Example: Show that

$$\mathbf{v} = \begin{pmatrix} 2i \\ -1 \end{pmatrix}$$

is an eigenvector of the matrix

$$A = \begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix}$$

corresponding to the eigenvalue $\lambda = 1 + 2i$.

$$A\mathbf{v} = \begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2i \\ -1 \end{pmatrix} = \begin{pmatrix} 2i-4 \\ -2i-1 \end{pmatrix}.$$

Also,

$$\lambda\mathbf{v} = (1+2i) \begin{pmatrix} 2i \\ -1 \end{pmatrix} = \begin{pmatrix} 2i-4 \\ -1-2i \end{pmatrix}$$

and so, since $A\mathbf{v} = \lambda\mathbf{v}$, $1+2i$ is an eigenvalue of the matrix with the given eigenvector.

We will often want to compute the real and imaginary parts of complex-valued expressions such as the following.

Example: Find the real and imaginary parts of

$$e^{(1+2i)t} \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Using Euler's formula, this expression becomes

$$\begin{aligned} e^t(\cos 2t + i\sin 2t)\begin{pmatrix} i \\ 1 \end{pmatrix} &= e^t\begin{pmatrix} i\cos 2t - \sin 2t \\ \cos 2t + i\sin 2t \end{pmatrix} \\ &= e^t\begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix} + ie^t\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} \end{aligned}$$

Thus, the real part of the expression is

$$e^t\begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix}$$

and the imaginary part is

$$e^t\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}.$$

Exercise: Show that the real and imaginary parts of

$$e^{(4-i)t}\begin{pmatrix} 1 \\ 2-3i \\ -i \end{pmatrix}$$

are, respectively,

$$e^{4t}\begin{pmatrix} \cos t \\ 2\cos t - 3\sin t \\ -\sin t \end{pmatrix} \quad \text{and} \quad -e^{4t}\begin{pmatrix} \sin t \\ 2\sin t + 3\cos t \\ \cos t \end{pmatrix}.$$

8 Some exercises

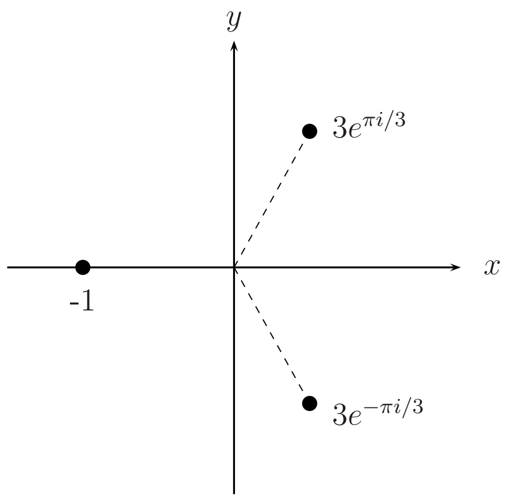
1. Find all of the solutions to the equation $z^3 + 27 = 0$ and plot them on a graph.
2. Find all of the solutions to the equation $z^4 + 81 = 0$ and plot them on a graph.
3. Calculate $\dfrac{1+5i}{3-2i}$.
4. Calculate the polar forms of $z_1 = \sqrt{3} + i$ and $z_2 = 1 + i$ and plot z_1 and z_2 on a graph. Calculate the polar forms of $z_1 z_2$ and z_1/z_2 and show these complex numbers on the same graph. Show the arguments and absolute values (moduli) of these complex numbers on your graph.

For further exercises, see the lecture handout "Exercises using complex numbers".

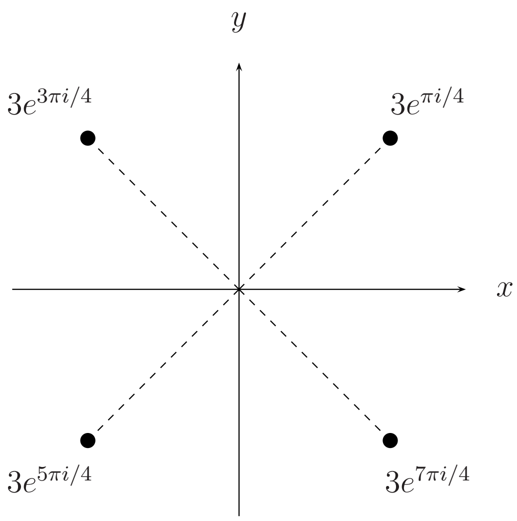
9 Solutions to exercises

1. $z^3 = -27$ so $|z|^3 = 27$ and thus $|z| = 3$. Hence we can write $z = 3e^{i\theta}$. Plugging this into the equation gives $e^{3i\theta} = -1$. Hence $3\theta = \pi + 2n\pi$, where n is an integer. Hence $\theta = \pi/3 + 2n\pi/3$, $n = 0, 1, 2$. There are no more solutions because starting at $n = 3$, the previous solutions reappear. Note that you could describe these solutions with negative values of θ as well (e.g. take $n = 1, -1, 0$). The important thing is to get the correct complex numbers which are

$$\begin{aligned} z &= -1, \quad \text{for } n = 1, \\ z &= 3e^{\pi i/3} = \frac{3}{2}(1 + i\sqrt{3}), \quad \text{for } n = 0, \\ z &= 3e^{-\pi i/3} = \frac{3}{2}(1 - i\sqrt{3}), \quad \text{for } n = -1. \end{aligned}$$



2. $z^4 = -81$ so $|z|^4 = 81$ and thus $|z| = 3$. Hence we can write $z = 3e^{i\theta}$. Plugging this into the equation gives $e^{4i\theta} = -1$. Hence $4\theta = \pi + 2n\pi$, where n is an integer. Hence $\theta = \pi/4 + n\pi/2$, $n = 0, 1, 2, 3$. There are no more solutions because starting at $n = 4$, the previous solutions reappear. Note that you could describe these solutions with negative values of θ as well (e.g. take $n = -2, -1, 0, 1$). The important thing is to get the correct complex numbers as shown on the graph below.



3.

$$\frac{1+5i}{3-2i} = \frac{(1+5i)}{(3-2i)}\frac{(3+2i)}{(3+2i)} = \frac{3-10+15i+2i}{3^2+2^2} = -\frac{7}{13}+\frac{17}{13}i.$$

4. $|z_1| = \sqrt{(3+1)} = 2$. Hence $z_1 = 2e^{i\theta_1}$, where $\theta_1 = \tan^{-1}(1/\sqrt{3}) = \pi/6$. Similarly, $z_2 = \sqrt{2}e^{i\pi/4}$. $z_1 z_2 = 2\sqrt{2}e^{i(\pi/6+\pi/4)} = 2\sqrt{2}e^{5\pi i/12}$. Also, $z_1/z_2 = \sqrt{2}e^{i(\pi/6-\pi/4)} = \sqrt{2}e^{-\pi i/12}$. Of course you could calculate these without using the polar form as well. Here is the picture: