DEPARTMENT OF MATHEMATICS

MATHS 260 Exercises using complex numbers

The following exercises test your ability to use complex numbers in the ways necessary for Maths 260. Have a go at these - if you can get the right answers for all questions up to and including 3(e), then you will not need to come to Lectures 19, 20 and 21. The answers to the questions are on the back of the sheet.

Questions 5(f) and (g) do not use complex numbers, but you should be able to do them anyhow.

1. Let $z = 3 + 2i$, compute Re z, Im z, complex conjugate $\bar{z}$, $z + \bar{z}$, $z\bar{z}$, modulus $|z|$, $|\bar{z}|$ and z^2.

2. Calculate the following expressions.

   (a) $(-1 + 3i) + 0.5(2 + 2i)$ (b) $(3 + 2i) + (3 + i)$ (c) $(4 + 2i) - (3 - 2i)$

   (d) $(3 + 4i) - (2 - 3i)$ (e) $(3 + 2i)(3 + i)$ (f) $(4 - 2i)(3 - 2i)$

   (g) $\frac{1+3i}{2+i}$ (h) $\frac{-2+6i}{1+2i}$

3. Find the real and imaginary parts of the following expressions.

   (a) $(3 - 4i)(7i - 1)$ (b) $\exp[(2 + 3i)t]$ (c) $(4i - 2)\begin{pmatrix} 23 \\ 2i - 6 \end{pmatrix}$

   (d) $\exp(it)\begin{pmatrix} 1 \\ 2 - i \end{pmatrix}$ (e) $\exp[(-3 - i)t]\begin{pmatrix} 1 - 3i \\ 4i \end{pmatrix}$

4. Find the determinant of each of the following matrices.

   (a) $\begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 2 - \lambda & -3 \\ 1 & 4 - \lambda \end{pmatrix}$

   (c) $\begin{pmatrix} 1 - \lambda & 0 & 0 \\ 2 & 7 - \lambda & 0 \\ -2 & 3 & -1 - \lambda \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix}$

5. Find the eigenvalues and eigenvectors of the following matrices.

   (a) $\begin{pmatrix} -1 & 2 \\ -7 & 8 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 4 \\ -1 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}$

   (d) $\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 2 & -1 & 0 \\ 5 & 2 & 4 \\ 0 & 1 & 2 \end{pmatrix}$ (f) $\begin{pmatrix} 2 & -1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

   (g) $\begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Answers

1. Re $z = 3$, Im $z = 2$, $\bar{z} = 3 - 2i$, $z + \bar{z} = 6$, $z\bar{z} = 13$, modulus $|z| = \sqrt{13}$, $|\bar{z}| = \sqrt{13}$ and $z^2 = 5 + 12i$.

2.

(a) $4i$	(b) $6+3i$	(c) 1
(d) $1+7i$	(e) $7+9i$	(f) $8-14i$
(g) $1+i$	(h) $2+2i$	

3. Write $a \equiv \text{Re}z$, $b \equiv \text{Im}z$.

   (a) $a = 25, b = 25$

   (b) $a = e^{2t}\cos 3t$, $b = e^{2t}\sin 3t$

   (c) $a = \begin{pmatrix} -46 \\ 4 \end{pmatrix}$, $b = \begin{pmatrix} 92 \\ -28 \end{pmatrix}$

   (d) $a = \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix}$, $b = \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}$,

   (e) $a = e^{-3t}\begin{pmatrix} \cos t + 3\sin t \\ 4\sin t \end{pmatrix}$, $b = e^{-3t}\begin{pmatrix} -\sin t - 3\cos t \\ 4\cos t \end{pmatrix}$

4. (a) 11 (b) $\lambda^2 - 6\lambda + 11$ (c) $(1-\lambda)(7-\lambda)(-1-\lambda)$ (d) 1

5. Write $\mathbf{v}_j$ for the eigenvector corresponding to eigenvalue λ_j.

   (a) $\lambda_1 = 6$, $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$; $\lambda_2 = 1$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

   (b) $\lambda_1 = 2+2i$, $\mathbf{v}_1 = \begin{pmatrix} -2i \\ 1 \end{pmatrix}$; $\lambda_2 = 2-2i$, $\mathbf{v}_2 = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$

   (c) $\lambda_1 = 1+\sqrt{6}i$, $\mathbf{v}_1 = \begin{pmatrix} 2 \\ \sqrt{6}i \end{pmatrix}$; $\lambda_2 = 1-\sqrt{6}i$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -\sqrt{6}i \end{pmatrix}$

   (d) $\lambda_1 = 1+\sqrt{6}$, $\mathbf{v}_1 = \begin{pmatrix} 2 \\ \sqrt{6} \end{pmatrix}$; $\lambda_2 = 1-\sqrt{6}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -\sqrt{6} \end{pmatrix}$

   (e) $\lambda_1 = 2$, $\mathbf{v}_1 = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}$; $\lambda_2 = 2+i$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -i \\ -1 \end{pmatrix}$; $\lambda_3 = 2-i$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix}$

   (f) $\lambda_1 = 2$ (twice) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; $\lambda_3 = 3$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

   (g) $\lambda_1 = 2$ (twice) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (only one linearly independent eigenvector);

   $\lambda_3 = 3$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$