Maths 260 Lecture 4

- **Topics for today:**
  More on Euler's method
  Improved Euler's method
  4th-order Runge-Kutta method
- **Reading for this lecture:** BDH Sections 1.4, 7.1
- **Suggested exercises:** BDH Sect. 1.4, #1, Sect. 7.1, #6
- **Reading for next lecture:** BDH Sections 7.2-7.4
- **Today's handout:**
  Pictures from Lecture 4
  Tutorial 2

More on Euler's Method

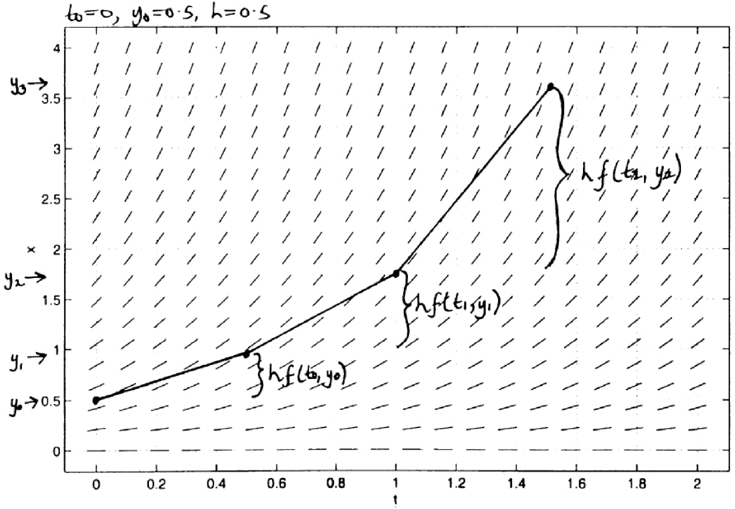
The main idea for Euler's method is as follows.

To approximate the solution to the IVP

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0$$

start at (t_0, y_0) and take small steps, with the direction of each step being the direction of the slope field at the start of that step.

The following picture illustrates the relationship between the slope field and the numerical solution obtained from Euler's method.



In the next example we can solve the IVP exactly and hence check the accuracy of Euler's method for various choices of step size.

Example 1: For the IVP

$$\frac{dy}{dt} = yt, \qquad y(0) = 1$$

calculate an approximation to y(0.4) using Euler's method with

(i) $h = 0.2$, and
(ii) $h = 0.1$.

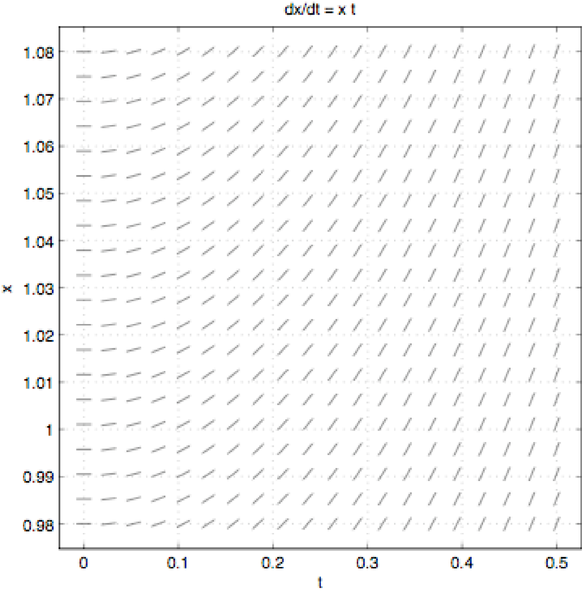
Calculate the error in each approximation.

For $h = 0.2$:

n	t_n	y_n	$f(t_n, y_n)$	$y_n + hf(t_n, y_n)$
0	0.0	1.0	0.0	1.0
1	0.2	1.0	0.2	1.04
2	0.4	1.04		

For $h = 0.1$:

n	t_n	y_n	$f(t_n, y_n)$	$y_n + hf(t_n, y_n)$
0	0.0	1.0	0.0	1.0
1	0.1	1.0	0.1	1.01
2	0.2	1.01	0.202	1.0302
3	0.3	1.0302	0.30906	1.061106
4	0.4	1.0611		


dx/dt = x t
x
1.08
1.07
1.06
1.05
1.04
1.03
1.02
1.01
1
0.99
0.98
0
0.1
0.2
0.3
0.4
0.5
t

To calculate the error in the approximation, we need to compare with the actual solution.

Exercise: Show that $y(t) = \exp(t^2/2)$ solves the IVP.

Using the explicit solution from the exercise, we get $y(0.4) = \exp(0.08) \approx 1.0833$.

Error in the first approximation (with $h = 0.2$) is:

Error in the second approximation (with $\mathrm{h} = 0.1$) is:

The error was approximately halved by halving the step size, but twice as many steps/calculations were done to obtain this improvement in accuracy.

When using Euler's method, picking a smaller step size will usually give a more accurate approximation - but will involve more work. We return to this idea in the next lecture.

Improving Euler's Method

For small h we have

$$\frac{y(t_{n+1}) - y(t_n)}{h} \approx \frac{dy}{dt} = f(t, y)$$

So

$$y(t_{n+1}) = y(t_n) + hf(t_n, y(t_n)) + \epsilon_n$$

where ϵ_n is the error made in the approximation.

Euler's Method approximates this formula by dropping ϵ_n from the equation, so that the Euler estimate at t_{n+1} is

$$y(t_{n+1}) = y(t_n) + hf(t_n, y(t_n))$$

Geometrically, Euler's method amounts to following a tangent line, instead of the (unknown) solution curve, from y_n to the value we accept for y_{n+1}.

The direction of each step is determined by the slope at the beginning of the step.

Since the slope of the actual solution curve varies throughout the interval from t_n to t_{n+1}, the value of y_{n+1} calculated by Euler's method generally does not agree with the value on the solution curve.

We can obtain a more accurate method by adjusting the direction of the step according to the slope field seen along an Euler step.

Improved Euler's method (IE)

To take one step of length h with Improved Euler's method:

(a) Take an ordinary Euler step of length h. Calculate the slope at the end of this step.

(b) Go back to the beginning of the step, take a step of length h with slope being the average of the slope at the beginning of the step and the slope calculated in (a).

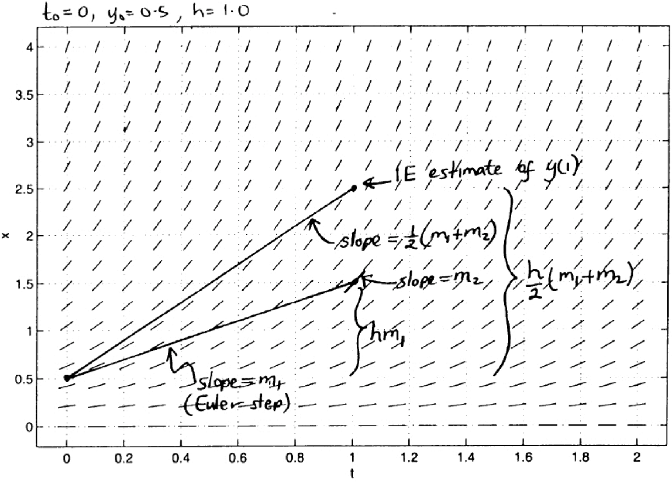
The formulas for this method are

$$t_{n+1} = t_n + h$$
$$y_{n+1} = y_n + \frac{h}{2}(m_1 + m_2)$$

where

$$m_1 = f(t_n, y_n)$$
$$m_2 = f(t_{n+1}, y_n + hf(t_n, y_n))$$

The following picture illustrates the relationship between the slope field and the numerical solution obtained with the IE method.



Example 2: Use $h = 0.5$ in the IE method to calculate an approximation to the solution of the IVP

$$\frac{dy}{dt} = -2ty^2, \qquad y(0) = 1$$

at $t = 1.0$.

Using the routine *numerical* from MATLAB, we can see how changing the step size in the IE method improves the solution:

Output from *numerical* for the above IVP, finding the y value at t=1.0, is:

No. of Steps	Stepsize (h)	approx. $y(1)$
1	1.0	0.0000000
2	0.5	0.4995117
4	0.25	0.5048106
8	0.125	0.5014094
16	0.0625	0.5003669

Using separation of variables, we can calculate the true solution at $t = 1$, i.e., $y(1.0) = 0.5$.

Notice that accuracy of the numerical solution is improved when a smaller step size is used.

4th-order Runge-Kutta method (RK4)

RK4 is the most commonly used fixed step size numerical method for IVPs.

This method evaluates the slope $f(t, y)$ four times within each step. Starting at (t_n, y_n) we calculate (t_{n+1}, y_{n+1}) as follows:

$$t_{n+1} = t_n + h$$

$$m_1 = f(t_n, y_n)$$

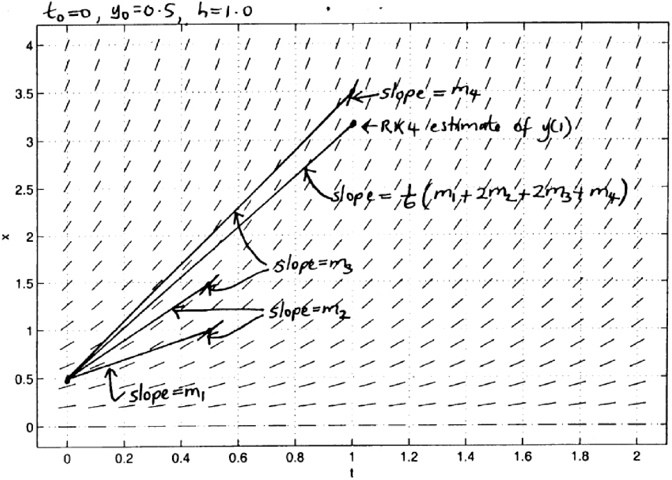
$$m_2 = f(t_n + \tfrac{h}{2}, y_n + \tfrac{h}{2} m_1)$$

$$m_3 = f(t_n + \tfrac{h}{2}, y_n + \tfrac{h}{2} m_2)$$

$$m_4 = f(t_n + h, y_n + h m_3)$$

$$y_{n+1} = y_n + \tfrac{h}{6}\left(m_1 + 2m_2 + 2m_3 + m_4\right)$$

The following picture illustrates the relationship between the slope field and the numerical solution obtained with RK4.



Example 3: Use $h = 0.5$ and one step of RK4-method to calculate an approximation to the solution of the IVP

$$\frac{dy}{dt} = -2ty^2, \qquad y(0) = 1$$

at $t = 0.5$.

Important ideas from today:

- Numerical methods approximate solutions to IVPs.
- Euler's method uses the slope at the beginning of each step. Better methods adjust the direction of each step according to the slope field seen along an Euler step.
- The error in a numerical approximation generally reduces if the step size is decreased - but using smaller steps means more work.