Maths 260 2009F - Solutions to mid-semester test

Q1 (a) This is a separable DE.

$$\frac{dy}{dt} = \frac{-t}{y} \Rightarrow \int y\,dy = \int -t\,dt$$

$$\Rightarrow \frac{y^2}{2} + c_1 = -\frac{t^2}{2} + c_2$$

$$\Rightarrow y^2 = -t^2 + k$$

$$k = 2(c_2 - c_1)$$

$$\Rightarrow y(t) = \pm\sqrt{k - t^2}$$

(b) $y(1) = -2 \Rightarrow$ we use the negative square root in the expression for $y(t)$.

Then $-2 = -\sqrt{k - 1^2}$

$$\Rightarrow (-2)^2 = \left(-\sqrt{k-1}\right)^2$$

$$\Rightarrow 4 = k - 1$$

$$\Rightarrow k = 5$$

so $y(t) = -\sqrt{5 - t^2}$ is the solution to the IVP.

(c) This solution is defined if $5 - t^2 \geq 0$

ie. $-\sqrt{5} \leq t \leq \sqrt{5}$

[NB however that at $t = \pm\sqrt{5}$, $y = 0$ and so the DE is not properly defined.]

Q2 (a) $\dfrac{dy}{dt} = -3y - y^2 = -y(3+y)$

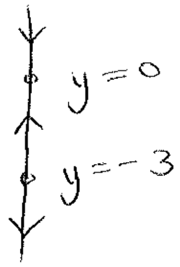
Equilibria are $y=0$ and $y=-3$.

$\dfrac{\partial f}{\partial y} = -3 - 2y \qquad (f(y) = -3y - y^2)$

At $y=0$, $\dfrac{\partial f}{\partial y} = -3 \Rightarrow y=0$ is a sink.

At $y=-3$, $\dfrac{\partial f}{\partial y} = -3-2(-3) = 3 \Rightarrow y=0$ is a source

Phase line:

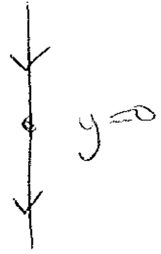


(b) $\dfrac{dy}{dt} = -y^2 = f(y)$

Equilibrium is $y=0$

$\dfrac{\partial f}{\partial y} = -2y = 0$ at $y=0$

Thus linearisation does not tell us the type of the equilibrium at $y=0$. However, $-y^2 \leq 0$ for all y so $\dfrac{dy}{dt} \leq 0$ for all $y \Rightarrow y$ decreases with time except at $y=0$. The phase line is, therefore:



Thus, $y=0$ is a node.

(c) $\frac{dy}{dt} = 3y - y^2 = y(3-y)$

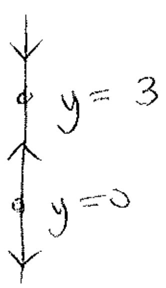
Equilibria are $y=0$, $y=3$.

$\frac{\partial f}{\partial y} = 3 - 2y$

At $y=0$, $\frac{\partial f}{\partial y} = 3 \Rightarrow y=0$ is a source.

At $y=3$, $\frac{\partial f}{\partial y} = -3 \Rightarrow y=3$ is a sink.

Hence the phase line is:



(d) $\frac{dy}{dt} = ay - y^2 = y(a-y)$.

Equilibria are at $y=0$ and $y=a$.

$\frac{\partial f}{\partial y} = a - 2y$

At $y=0$, $\frac{\partial f}{\partial y} = a \Rightarrow y=0$ is a sink if $a<0$

$y=0$ is a source if $a>0$

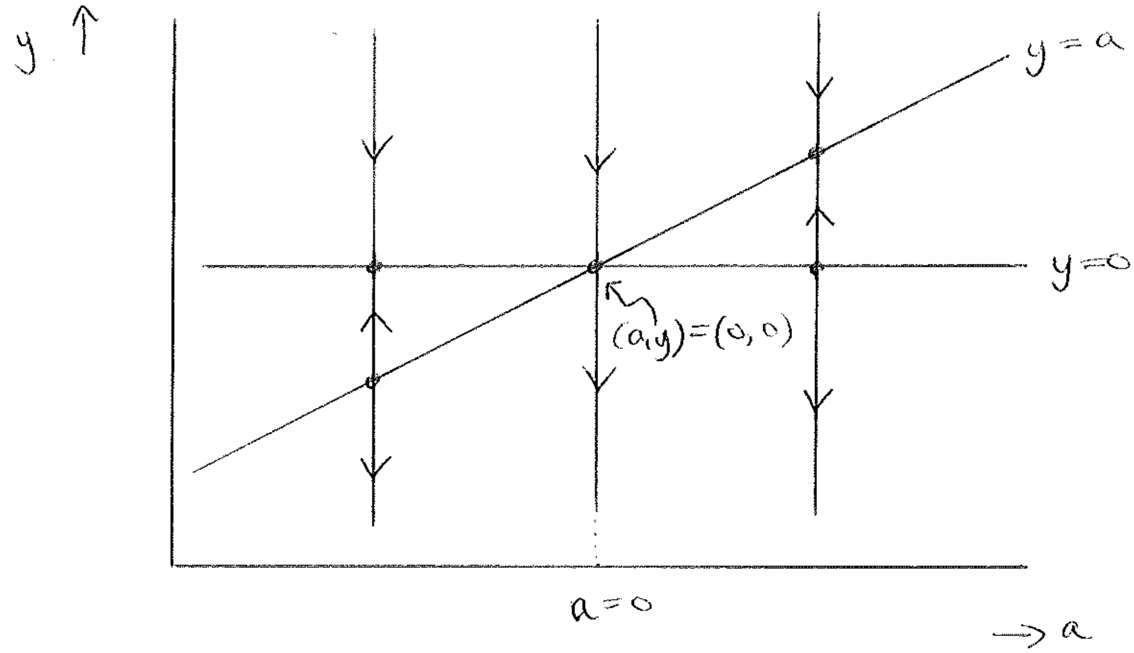
If $a=0$, $y=0$ is a node, from part (b).

At $y=a$, $\frac{\partial f}{\partial y} = -a \Rightarrow y=a$ is a sink if $a>0$

$y=a$ is a source if $a<0$

If $a=0$, $y=a$ is a node, from part (b)

Bifurcation diagram



Bifurcation is at $a = 0$.

Q3 (a) $A = \begin{pmatrix} -2 & 3 \\ 0 & -1 \end{pmatrix}$ has eigenvalues -2 and -1 with eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ respectively.

The straightline solns are therefore

$$Y(t) = c_1 e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad Y(t) = c_2 e^{-t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

(b) General solution:

$$Y(t) = c_1 e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

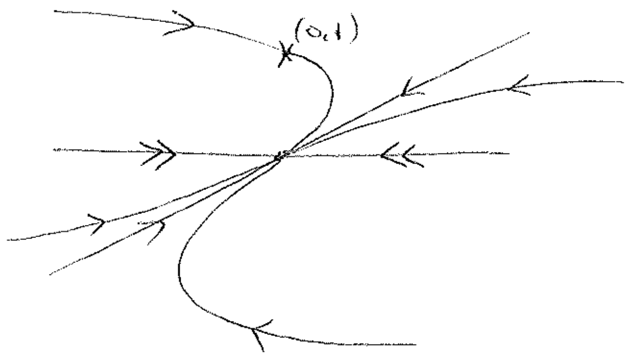
(c) $Y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\Rightarrow \begin{cases} c_1 + 3c_2 = 0 \\ c_2 = 1 \end{cases} \qquad \text{ie. } c_1 = -3, \quad c_2 = 1$$

so $\quad Y(t) = -3e^{-2t}\begin{pmatrix}1\\0\end{pmatrix} + e^{-t}\begin{pmatrix}3\\1\end{pmatrix}$

or $\quad (x(t), y(t)) = \left(3e^{-t} - 3e^{-2t},\ e^{-t}\right)$

(d)



(e) As $t \to \infty$ all solns tend to $(0,0)$.

Q4 $\quad \dfrac{dc}{dt} = kc(100 - c) - s = 100kc - kc^2 - s$

(a) The term $100kc$ represents growth of the population in the absence of s (ie. when $s=0$) and when $c \ll 100$. This means the population grows linearly when the population is small enough (and for $s=0$).

The term $-kc^2$ is a logistic term. It ensures that the population cannot grow forever. For instance, if $s=0$, then the population will decrease if $c > 100$ because of this term. The term models competition between cockroaches (e.g. for food, space, etc)

The term $-s$ models some negative effect that is independent of the size of the population. For instance this term might model the effect of a cockroach trap that removes s (or, better, $100s$) cockroaches from the restaurant per week.

(b) If $c << 100$ and $s=0$, $\frac{dc}{dt} \approx 100kc$

$$\frac{dc}{dt} = 0.1c \implies 100k = 0.1$$

or $k = 10^{-3}$

(c) Equilibrium solns satisfy

$$100kc - kc^2 - s = 0$$

$$c = \frac{-100k \pm \sqrt{(100k)^2 - 4(-s)(-k)}}{-2k}$$

$$= 50 \pm \frac{\sqrt{(100k)^2 - 4ks}}{2k}$$

The maximum value occurs for $s=0$, in which case $c = 100$. This means the maximum possible population of cockroaches is 100,000.
($c=1 \implies 1000$ cockroaches).

Q5 (a) $f(t,y) = \frac{y^2}{t} + t$

$$\frac{\partial f}{\partial y} = \frac{2y}{t}$$

Both f and $\frac{\partial f}{\partial y}$ are continuous functions of y, and both are continuous functions of t except at $t=0$. The initial condition is $t=2$, $y=0$. Since both f and $\frac{\partial f}{\partial y}$ are continuous at $t=2$, $y=0$, the Existence and Uniqueness Theorems guarantee that the IVP has a unique soln.

(b). Any initial condition with $t=0$ violates the hypotheses of the Existence and Uniqueness Theorems. So, for instance, $y(0) = 1$ is an initial condition for which the IVP is not ~~guarante~~ guaranteed to have a unique solution.