THE UNIVERSITY OF AUCKLAND

SECOND SEMESTER, 2005
Campus: City

MATHEMATICS

Differential Equations

(Time allowed: THREE hours)

NOTE: Answer **ALL** questions. Show **ALL** your working. 100 marks in total.

1. (10 marks)

(a) Find the general solution of the differential equation

$$t^2\frac{dy}{dt} + ty = t - t^2, \quad t > 0.$$

(b) Solve the initial value problem

$$\frac{dy}{dt} = y^3 e^{2t}, \quad y(0) = -0.5.$$

2. (10 marks)
Consider the initial value problem

$$\frac{dy}{dt} = t^2 - y^2, \quad y(1) = 0.$$

(a) Use existence and uniqueness theorems to show there exists a unique solution to this initial value problem.
(b) Use two steps of the Improved Euler method to approximate $y(1.4)$.
(c) Explain what is means to say that the order of the Improved Euler method is 2.

3. (15 marks)
Consider the following two systems of differential equations:

$$(a) \quad \frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{Y} \qquad (b) \quad \frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix}\mathbf{Y}$$

(i) Determine the general solution for each system.
(ii) Carefully sketch the phase portrait for each system.
(iii) Describe the long term behaviour for each system if $\mathbf{Y}(0) = (0, 8)^T$.

4. (12 marks)
Consider the one-parameter family of differential equations

$$\frac{dy}{dt} = y(y^2 + \lambda),$$

where λ is the parameter.

(a) Find all the equilibrium solutions, and determine their type.

(b) For $\lambda = -1$, draw the phase line and use it to sketch the solutions in the $t - y$ plane showing the long term behaviour.

(c) Sketch the bifurcation diagram and find the bifurcation value for this family of differential equations.

5. (24 marks)
Two populations living in the same area are modelled by the differential equation system

$$\frac{dx}{dt} = x(1 - 0.1x) - 0.05xy$$
$$\frac{dy}{dt} = y(1.7 - 0.1y) - 0.15xy$$

where the populations are measured in **thousands** and the time in years.

(a) Explain the physical significance of the terms $-0.05xy$ and $-0.15xy$.

(b) Find all the equilibrium solutions of the system.

(c) Find the Jacobian at the equilibrium solutions, and determine their type (saddle, sink, source etc).

(d) On the attached grid, sketch the phase portrait for the system by drawing the nullclines for the system, showing the direction of the vector field between the nullclines and on the nullclines, marking in the equilibrium solutions, and sketching some solutions including those through $(2, 2)$ and $(4, 18)$.

(e) If there are initially 10,000 of the population represented by x, and 4,000 of the population represented by y, describe the behaviour of the two populations.

6. (7 marks)
Consider the following differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = \sin t.$$

(a) Find the general solution.

(b) Describe the long term behaviour of the solution.

7. (7 marks)
Solve the following initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = t, \quad y(0) = 1, y'(0) = 0.$$

8. (15 marks)
In this question, you may use the table of Laplace transforms attached.

(a) If

$$h(t) = \begin{cases} 0, & t < 4 \\ t-4, & t \geq 4, \end{cases}$$

show that

$$\mathcal{L}\{h(t)\} = \frac{e^{-4s}}{s^2}.$$

(b) Show that

$$\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s^2(s^2+4)}\right\} = \frac{1}{8}(2(t-4) - \sin 2(t-4))\mathcal{U}_4(t),$$

where

$$\mathcal{U}_4(t) = \begin{cases} 0, & t < 4 \\ 1, & t \geq 4. \end{cases}$$

(c) Use the method of Laplace transforms and your answers to (a) and (b) to find a solution to the initial value problem

$$\frac{d^2y}{dt^2} + 4y = h(t), \quad y(0) = 0, \quad y'(0) = 1,$$

where $h(t)$ is as defined in (a).

A brief table of Laplace transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\dfrac{1}{s}$
e^{at}	$\dfrac{1}{s-a}$
$t^n,\ n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}$
$e^{at}t^n,\ n = 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}$
$\sin(bt)$	$\dfrac{b}{s^2+b^2}$
$\cos(bt)$	$\dfrac{s}{s^2+b^2}$
$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
$\mathcal{U}_a(t),\ a \geq 0$	$\dfrac{e^{-as}}{s}$
$f(t-a)\mathcal{U}_a(t),\ a \geq 0$	$e^{-as}F(s)$
$e^{at}f(t)$	$F(s-a)$
$\dfrac{d^n f}{dt^n}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{n-1}(0)$

ANSWER SHEET FOLLOWS

Candidate's Name: ______________________________ ID No: ______________

TIE THIS ANSWER SHEET TO YOUR SCRIPT BOOK

Answer sheet for Question 5(d)