Maths 260 Lecture 12

- **Topics for today:**
  Linear equations
- **Reading for this lecture:** BDH Section 1.8
- **Suggested exercises:** BDH Section 1.8, #1,3,9,11,13,15
- **Reading for next lecture:** BDH Section 1.9
- **Today's handouts:** None

Linear Differential Equations

A first order DE is **linear** if it can be written in the form

$$\frac{dy}{dt} = a(t)y + b(t)$$

where $a(t)$ and $b(t)$ are arbitrary functions of t.

Saying the equation is linear means that the dependent variable, y, appears in the equation to the first power only.

Example 1: Which of the following equations is linear?

- $\dfrac{dy}{dt} = y \cos t + t^2$
- $y\dfrac{dy}{dt} = ty^2 + ty$
- $\dfrac{dP}{dt} = e^t P + \sin t$
- $\dfrac{dy}{dt} = ty(1 - y)$
- $\dfrac{dy}{dt} = ty$

Some terminology

A linear differential equation of the form

$$\frac{dy}{dt} = a(t)y + b(t)$$

is said to be **homogeneous** or **unforced** if $b(t) = 0$ for all t. Otherwise, the DE is **nonhomogeneous** or **forced.**

The DE above is called a **constant coefficient** equation if $a(t)$ is a constant, i.e., if the DE has the form

$$\frac{dy}{dt} = \lambda y + b(t)$$

where λ is a constant.

Note that a homogeneous linear equation is separable.

Nice properties of linear DEs

The linearity principle: If $y_h(t)$ is a solution of the homogeneous DE

$$\frac{dy}{dt} = a(t)y$$

then any constant multiple of $y_h(t)$ is also a solution, i.e., $ky_h(t)$ will be a solution for all choices of k.

Note that $y(t) = 0$ is a solution to every homogeneous DE.

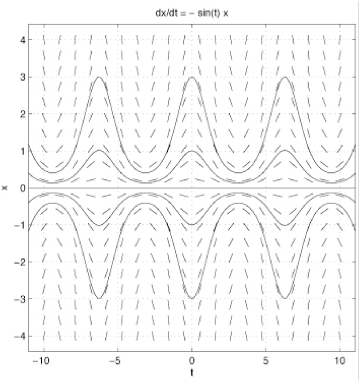
Example 2: The function

$$y(t) = ke^{\cos t}$$

is a solution to the homogeneous equation

$$\frac{dy}{dt} = (-\sin t)y$$

for all choices of the constant k. The slope field and some of these solutions (for various k) are plotted in the figure below.



The extended linearity principle:

Consider the nonhomogeneous equation

$$\frac{dy}{dt} = a(t)y + b(t)$$

and its associated homogeneous equation

$$\frac{dy}{dt} = a(t)y$$

If $y_h(t)$ is any solution of the homogeneous equation and $y_p(t)$ is any solution of the nonhomogeneous equation then $y_h(t) + y_p(t)$ is also a solution of the nonhomogeneous equation.

If $y_p(t)$ and $y_q(t)$ are two solutions of the nonhomogeneous equation then $y_p(t) - y_q(t)$ is a solution to the homogeneous equation.

Lastly, if $y_h(t)$ is nonzero, then $ky_h(t) + y_p(t)$ is a solution to the nonhomogeneous equation for all choices of k, and every solution of the nonhomogeneous equation can be written in this form by picking a suitable value of k.

We call the one-parameter family of solutions

$$y(t) = ky_h(t) + y_p(t)$$

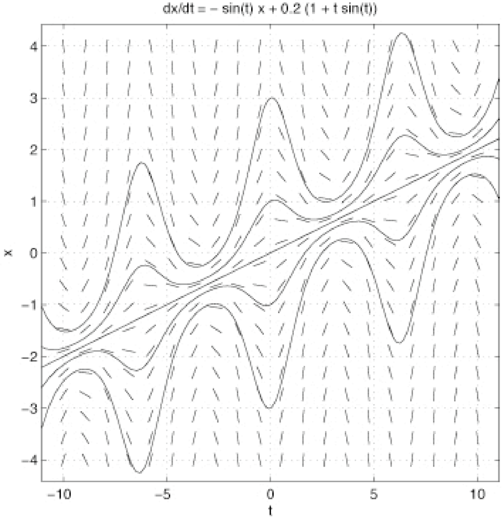
the **general solution** to the nonhomogeneous equation, because every solution of the equation can be expressed in this form.

Example 3: Consider the DE

$$\frac{dy}{dt} = (-\sin t)y + \frac{1}{5}(1 + t\sin t)$$

Show that $y_p(t) = t/5$ is a solution to this differential equation and hence write down the general solution.

The slope field and graphs of various solutions to the DE are shown below.



Solving linear differential equations

We can now, in principle, solve linear DEs by the following method:

- Find the general solution to the associated homogeneous DE.
- Find one particular solution to the nonhomogeneous DE.
- Obtain the general solution to the nonhomogeneous DE by adding the general solution to the homogeneous DE and the particular solution to the nonhomogeneous DE.

The first step is straightforward (use separation of variables) but the second step can be hard – usually we just guess and hope to be lucky.

Guessing a solution to a nonhomogeneous linear DE

Example 4: Consider the nonhomogeneous DE

$$\frac{dy}{dt} = -3y - e^t$$

The associated homogeneous equation is

$$\frac{dy}{dt} = -3y$$

which has the general solution $y(t) = ke^{-3t.}$

We rewrite the nonhomogeneous equation as

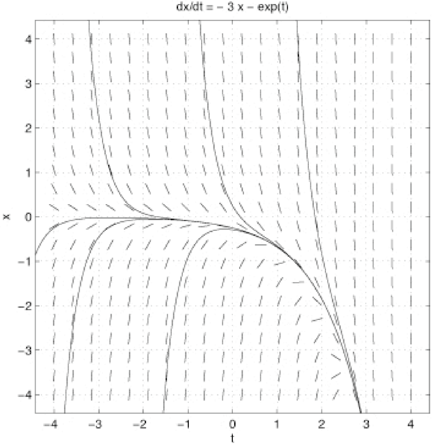
$$\frac{dy}{dt} + 3y = -e^t$$

and try to guess a solution. We want a function $y_p(t)$ that, when we substitute it into the left hand side of the DE, produces $-e^t$.

We find the general solution to the nonhomogeneous DE is

$$y(t) = ke^{-3t} - \frac{1}{4}e^t$$

for arbitrary k. The slope field for the DE and various solutions (for various choices of k) are plotted below.



Example 5: Now consider the nonhomogeneous DE

$$\frac{dy}{dt} + 3y = \sin 2t$$

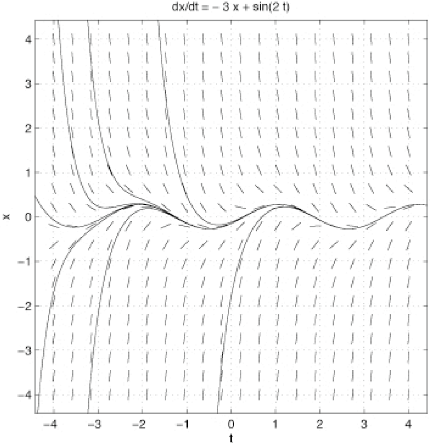
The general solution to the associated homogeneous equation is again $y(t) = ke^{-3t}$.

We want a function $y_p(t)$ that, when we substitute it into the left hand side of the DE produces $\sin 2t$.

The general solution to the nonhomogeneous DE is now

$$y(t) = ke^{-3t} + \frac{3}{13}\sin 2t - \frac{2}{13}\cos 2t$$

for arbitrary k. The slope field for the DE and various solutions (for various choices of k) are plotted below.



Example 6: Consider the nonhomogeneous DE

$$\frac{dy}{dt} + 3y = e^{-3t}$$

Now we want a function $y_p(t)$ that, when we substitute it into the left hand side of the DE produces $e^{-3t.}$

The basic idea with guessing solutions is to try $y_p(t)$ of the form of the forcing term $b(t)$.

So, if $b(t)$ is an exponential, try an exponential for $y_p(t)$. If $b(t)$ is a cosine or sine, try cosines and sines for $y_p(t)$. And so on.

If guessing does not produce a suitable $y_p(t)$, then we can try solving by integrating factors (see next lecture) or other methods (see later in course).

Important ideas from today:

- A first order DE is **linear** if it can be written in the form

$$\frac{dy}{dt} = a(t)y + b(t)$$

  where $a(t)$ and $b(t)$ are arbitrary functions of t.
- We discussed the linearity and extended linearity principles.
- We found solutions to some nonhomogeneous DEs by guessing the form of a solution and using substitution to determine coefficients.