THE UNIVERSITY OF AUCKLAND

FIRST SEMESTER, 2004
Campus: City

MATHEMATICS

Differential Equations

(Time allowed: THREE hours)

NOTE: Answer all questions. Show all your working. 100 marks in total.

1. (15 marks)

(a) Solve the initial-value problem

$$\frac{dy}{dt} = (t + \sin t)y^2, \qquad y(0) = 1.$$

(b) For the initial value problem

$$\frac{dy}{dt} = 3y^{2/3}, \qquad y(0) = 0,$$

$y(t) = t^3$ and $y(t) = 0$ are both solutions (do **not** show this). Explain why this does not contradict Uniqueness Theorem.

(c) Solve the initial-value problem

$$\frac{dy}{dt} = 2y + e^{3t}, \qquad y(0) = 1.$$

2. (10 marks) Find and classify the equilibrium points of

$$\frac{dy}{dt} = y^2 - ay$$

as a function of a. Sketch the bifurcation diagram.

3. (10 marks) Consider the initial value problem

$$\frac{dy}{dt} = \frac{2y}{t}, \qquad y(1) = 1.$$

(a) Show that the exact solution is $y(t) = t^2$.

(b) Use Euler's method to approximate the solution at $t = 3$, using step sizes of $h = 1$ and $h = 2$.

(c) Compute the error for each solution calculated in (b).

(d) Estimate (but do **not** calculate) the error you would obtain if you used Euler's method with $h = 0.5$ to solve the initial value problem.

4. (10 marks) Consider the predator-prey system of differential equations:

$$\frac{dx}{dt} = 0.2x - 0.1xy$$
$$\frac{dy}{dt} = -0.2y + 3xy$$

(a) Describe briefly the physical effect being modelled by each term in the system of equations.

(b) Describe the behaviour of the population of type x if population y is extinct. (You should sketch the phase line for the x population assuming that $y = 0$, and sketch the graphs of the x population as a function of time for several solutions. Then interpret these graphs for the x population.)

(c) How could you modify the system of equations to model the effect of species y being killed by hunters.

5. (15 marks) Consider the following two systems of differential equations:

(a) $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix} \mathbf{Y}$ (b) $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} \mathbf{Y}$

(i) Determine the general solution for each system. Express your answers in terms of real-valued functions.

(ii) Carefully sketch the phase portrait for each system.

(iii) Describe the long term behaviour of solutions in each system.

6. (10 marks)

(a) Find the general solution to the following differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 2e^{-t}.$$

(b) Solve the following initial value problem

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 8y = 0, \quad y(0) = 0, \ \frac{dy}{dt}(0) = 1.$$

7. (20 marks) Consider the following system of equations:

$$\begin{aligned} \frac{dx}{dt} &= -x(y+1) \\ \frac{dy}{dt} &= -x - y + a \end{aligned}$$

where a is a real constant.

A grid is provided on the yellow answer sheet attached to the back of the question paper. Use the grid for your answer to part (b) of this question. Attach the yellow answer sheet to your answer book.

(a) Show that there are two equilibrium solutions and that one of these is the point $(0, a)$. Determine the type (e.g., saddle, spiral source) of the equilibrium point at $(0, a)$.

(b) For $a = 1$ the equations become

$$\begin{aligned} \frac{dx}{dt} &= -x(y+1) \\ \frac{dy}{dt} &= -x - y + 1 \end{aligned}$$

(i) Determine the type (e.g., saddle, spiral source) of the other equilibrium solution (i.e., not the equilibrium solution at $(0, 1)$).

(ii) Find the nullclines for the system and sketch them on the answer sheet provided. Show the direction of the vector field in the regions between the nullclines and on the nullclines themselves.

(iii) Sketch the phase portrait for the system. Include in your phase portrait the solution curves passing through the initial conditions

A. $(x(0), y(0)) = (2, -2)$;

B. $(x(0), y(0)) = (-2, 2)$.

Make sure you show clearly where these solution curves go as $t \to \infty$.

8. (10 marks) In answering this question you may use the table of Laplace transforms attached to this exam paper.

(a) Find the inverse Laplace transform of

$$\frac{1}{(s+1)^2}.$$

(b) Use the method of Laplace transforms to find a solution to the initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 3, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = 1.$$

Hint: You may use without proof the result that

$$\frac{1}{s(s+1)^2} = \frac{1}{s} + \frac{-1}{s+1} + \frac{-1}{(s+1)^2}.$$

A brief table of Laplace transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\dfrac{1}{s}$
e^{at}	$\dfrac{1}{s-a}$
$t^n,\ n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}$
$e^{at}t^n,\ n = 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}$
$\sin(bt)$	$\dfrac{b}{s^2+b^2}$
$\cos(bt)$	$\dfrac{s}{s^2+b^2}$
$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
$\mathcal{U}_a,\ a \geq 0$	$\dfrac{e^{-as}}{s}$
$f(t-a)\mathcal{U}_a,\ a \geq 0$	$e^{-as}F(s)$
$e^{at}f(t)$	$F(s-a)$
$\dfrac{d^n f}{dt^n}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{n-1}(0)$

Candidate's Name: ______________________________ ID No: ______________

TIE THIS ANSWER SHEET TO YOUR SCRIPT BOOK

Answer sheet for Question 7(b)