# Maths 260 Lecture 30

- **Topic for today:**
  Higher order differential equations
- **Reading for this lecture:**
  BDH Section 3.6
- **Suggested exercises:**
  BDH Section 3.6; 1, 3, 5, 7, 9, 11
- **Reading for next lecture:**
  BDH Section 3.6
- **Today's handout:** Tutorial 11

# Example 1: Modelling a mass/spring system

We wish to model the motion of an object that is attached to a spring and slides in a straight line on a table.

Let $y(t)$ be the position of the object at time $t$, with $y = 0$ corresponding to the spring being neither stretched nor compressed.

**Main idea from physics:** Newton's second law says

mass $\times$ acceleration $=$ sum of forces acting on the object.

Typical forces on the object that we might consider are

- $r(y)$, the restoring force (the spring does not like to be compressed or stretched);
- $f(v)$, frictional forces, where $v = \frac{dy}{dt}$;
- $g(t, y)$, external forcing.

Substituting into Newton's law, we get

$$m\frac{d^2y}{dt^2} = r(y) + f(v) + g(t, y)$$

where $m$ is the mass of the object attached to the spring.

A common case assumes

- linear restoring force, i.e. $r(y) = -ky$ for some constant $k > 0$;
- linear damping, i.e. $f(v) = -bv$ for some constant $b > 0$;
- no spatial dependence in the forcing, i.e. $g$ is a function of $t$ but not of $y$.

The first two assumptions are often valid if $y$ and $v = \frac{dy}{dt}$ remain small.

We can write this case as

$$\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{1}{m}g(t)$$

This is an example of a higher order differential equation, i.e. a DE involving derivatives of second or higher order.

Other examples of higher order DEs:

- $\dfrac{d^2\theta}{dt^2} + c_1 \dfrac{d\theta}{dt} + c_2 \sin\theta = 0$

- $\dfrac{d^3y}{dt^3} - 2y\left(\dfrac{d^2y}{dt^2}\right)^2 + \dfrac{dy}{dt} = \sin t$

- A higher order system of DEs:

$$\begin{aligned} \frac{dx}{dt} &= 2x + y \\ \frac{d^2y}{dt^2} + \frac{dx}{dt}\frac{dy}{dt} + 3x &= 0 \end{aligned}$$

We can usually convert a higher order DE into an equivalent system of first order DEs. To do so, define new dependent variables as in the following examples.

**Example 2:** Rewrite the following equation as an equivalent system of first order equations:

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

**Example 3:** Rewrite the following equation as an equivalent system of first order equations:

$$\frac{d^3x}{dt^3} + 2\left(\frac{dx}{dt}\right)^2 = \sin t$$

Saying that a system of DEs is **equivalent** to a higher order DE means that if we know a solution to the system we can find a solution to the higher order equation, and vice versa.

**Example 4:** The function

$$y_1(t) = \sin \sqrt{\frac{k}{m}} t$$

is a solution to

$$\frac{d^2 y}{dt^2} + \frac{k}{m} y = 0$$

The pair of functions

$$y_1(t) = \sin \sqrt{\frac{k}{m}} t, \quad v_1(t) \equiv \frac{dy_1}{dt} = \sqrt{\frac{k}{m}} \cos \sqrt{\frac{k}{m}} t$$

is a solution to the equivalent system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{k}{m} y \end{aligned}$$

To determine the behaviour of solutions of a higher order DE we can rewrite the DE as the equivalent first order system.

Then we can study the system using the numerical methods and qualitative techniques already learnt (e.g. sketching solutions via phase plane methods). We can also use results like the Existence and Uniqueness Theorem.

However, in some special cases, it is convenient to study the original higher order equation directly.

For example, convenient analytic techniques exist for solving linear higher order equations – we will see these techniques in the next few lectures.

# Linear, Constant Coefficient, Higher Order DEs

A differential equation of the form

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0$$

where all $a_i$ are constant, and $a_n \neq 0$, is called an $n$th order, linear, constant coefficient DE.

**Example 5:** The differential equation

$$\frac{d^2 y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$$

is a second order, linear, constant coefficient DE.

We could solve this by converting to a system, then finding eigenvalues and eigenvectors etc, but there is a short cut for solving equations of this form.

# Idea behind the shortcut:

For Example 5, the equivalent system is:

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y} = \begin{pmatrix} y \\ z \end{pmatrix}$$

We expect solutions of the form

$$\mathbf{Y}(t) = \mathrm{e}^{\lambda t} \mathbf{v}.$$

The first component of such a $\mathbf{Y}$ is

$$y(t) = c\mathrm{e}^{\lambda t}$$

where $c$ is a constant (the first entry in $\mathbf{v}$).

Hence, guess a solution to the higher order DE of the form

$$y(t) = \mathrm{e}^{\lambda t}$$

where $\lambda$ is to be determined.

Substitute this candidate solution into our DE:

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$$

This is exactly what we would have got by using eigenvalues and eigenvectors to solve the equivalent system directly.

The equivalent system is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y} = \begin{pmatrix} y \\ z \end{pmatrix}$$

which has eigenvalues -3 and -2 with associated eigenvectors

$$\begin{pmatrix} 1 \\ -3 \end{pmatrix}, \text{ and} \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

respectively. The general solution is

$$\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = c_1 e^{-3t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

which gives $y(t) = c_1 e^{-3t} + c_2 e^{-2t}$.

This "guessing" method is often quicker than converting to a system and solving.

**Example 6:** Find some linearly independent solutions of

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

**Example 7:** Find some linearly independent solutions of

$$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} - 6\frac{dy}{dt} = 0$$

# General result:

Consider the differential equation

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0$$

Let $y_1(t), y_2(t), \ldots, y_n(t)$ be $n$ linearly independent solutions of the DE. Then for arbitrary constants $c_i$,

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

is the **general solution** to the DE. Every solution to the DE can be written in this form by picking the $c_i$ appropriately.

**Example 8:** Find the general solution of

$$\frac{d^2y}{dt^2} - 5y = 0$$

# Important ideas from today:

- A higher order differential equation can usually be rewritten as an equivalent system of first order differential equations.
- Solutions can then be investigated using the methods (qualitative, analytic, numerical) already studied for systems.
- However, in the case of linear, constant coefficient higher order equations it is usually possible and quicker to find analytic solutions directly. The 'guessing' method we use will be formalised in the next lecture.