1. (10 marks)

(a) Find the general solution to the following differential equation

$$\frac{dy}{dt} = t + yt,$$

$$\int \frac{dy}{y+1} = \int t\,dt$$

$$\ln(y+1) = t^2/2 + c$$

$$y = Ae^{t^2/2} - 1$$

(b) Find the general solution to the following differential equation

$$\frac{dy}{dt} = -2y + e^t.$$

$$\frac{dy}{dt} + 2y = e^t$$

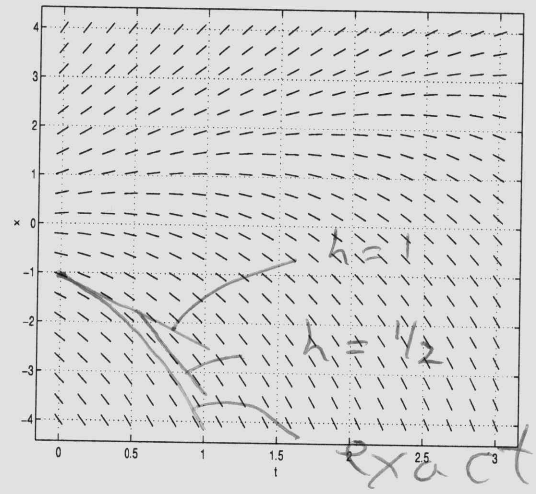
$$\cancel{\left(e^{2t}\frac{dy}{dt}\right)} \quad \frac{d}{dt}\left(ye^{2t}\right) = e^{3t}$$

$$ye^{2t} = \frac{1}{3}e^{3t} + c$$

$$y = \frac{1}{3}e^{t} + ce^{-2t}$$

2. (5 marks)

The following picture shows the slope field for a differential equation.



(a) On this picture, carefully draw the solution you would obtain if you used one step of Euler's method with $h = 1$ to approximate at $t = 1$ the solution to the differential equation satisfying the initial condition $x(0) = -1$.

(b) On the same picture, carefully draw the solution you would obtain if you used two steps of Euler's method with $h = 0.5$ to approximate the same solution.

(c) On the same picture draw the exact solution satistying the initial condition $x(0) = -1$. Use this to estimate the errors in the approximate solutions you obtained in (a) and (b) at $t = 1$.

$X(\text{exact}) = -4$

$X(_{h=1/2} = -3.2$

$X_{h=1} = -2.5$

Error $(h=1) = 1.5$

Error $(h=1/2) = 0.8$

3. (5 marks)
Consider the following initial value problem

$$\frac{dy}{dt} = y + \sin(t), \; y(0) = 1.$$

(a) Does a unique solution of the IVP exist? Give reasons for your answer.

(b) Use Improved Euler with stepsize $h = 1$ to find an approximation to the solution at $t = 1$.

(a) $f = y + \sin(t)$ is cts

& $\frac{\partial f}{\partial y} = 1$ is cts

$\Rightarrow$ unique soln for all y & t

(b) $m_1 = f(t_0, y_0) = 1 + \sin(0) = 1$

$m_2 = f(t_1, y_0 + h m_1)$

$= f(1, 1+1) = 2 + \sin(1) = 2.84$

$y_1 = y_0 + \frac{h}{2}(m_1 + m_2)$

$= 1 + \frac{1}{2}(1 + 2.841)$

$= 2.9207$

4. (10 marks)

Consider the following differential equation:

$$\frac{dy}{dt} = y^2 + y + \mu$$

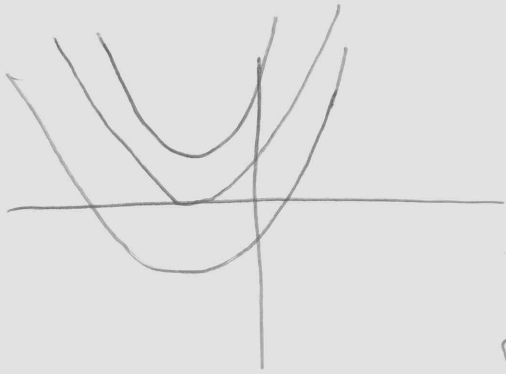
(a) Find all equilibrium solutions and determine their types (e.g., sink, node).

(b) Draw the bifurcation diagram. Identify any values of μ for which a bifurcation exists.

(a) $y^2 + y + \mu = 0$

$\Rightarrow \quad y = \dfrac{-1 \pm \sqrt{1-4\mu}}{2}, \quad \mu \leq 1/4$

From graph



~~smaller~~

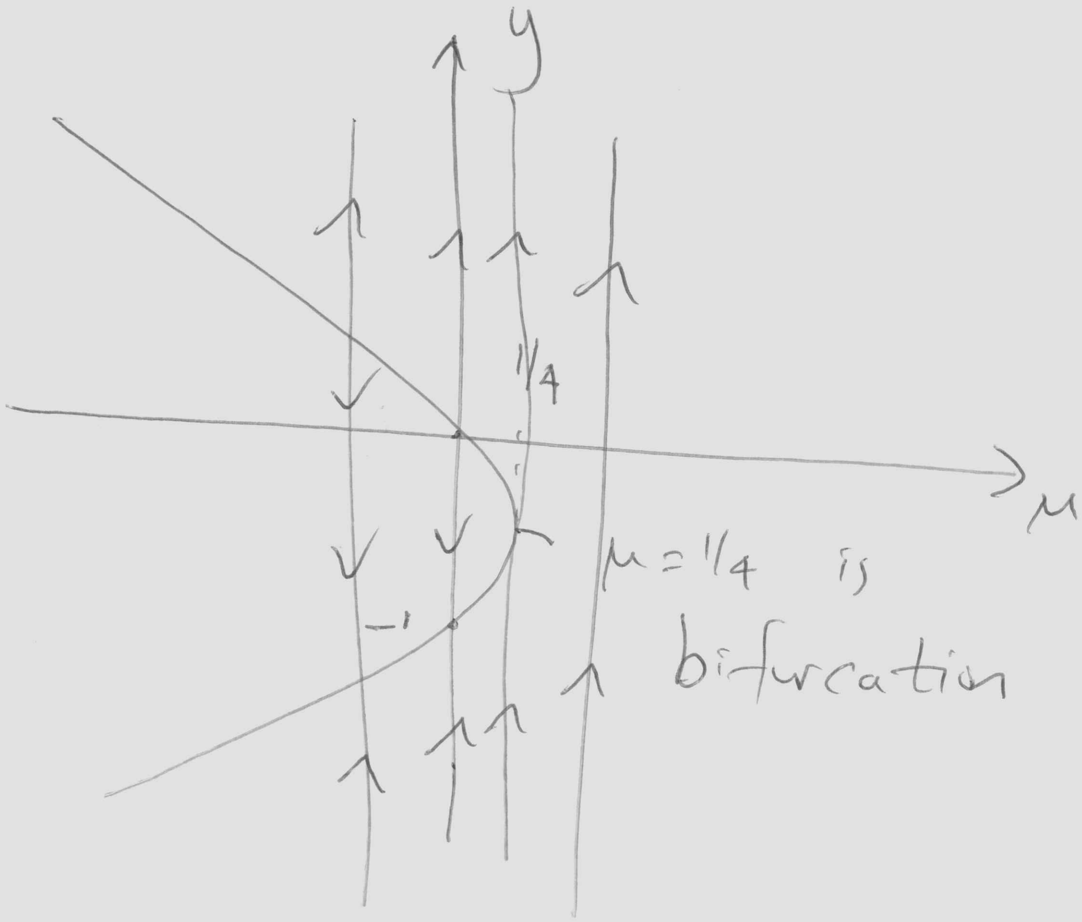
$\dfrac{-1 - \sqrt{1-4\mu}}{2}$ is sink & $\dfrac{-1 + \sqrt{1-4\mu}}{2}$ is a source.

or from $\dfrac{\partial f}{\partial y} = 2y + 1$

$\left.\dfrac{\partial f}{\partial y}\right|_{y = -\frac{1}{2} \pm \frac{\sqrt{1-4\mu}}{2}} = \pm\sqrt{1-4\mu}$ & obtain same result

(blank page for your working)

(b)



5. (10 marks)

Consider the following system of differential equations:

$$\frac{d}{dt}\begin{pmatrix} x \\ \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ \\ y \end{pmatrix}$$

(a) Find a solution to this system of equations. Your answer should contain two arbitrary constants.

(b) Find the solution that passes through $(x, y) = (1, 0)$ at $t = 0$. Express your solution in the form $(x(t), y(t))$.

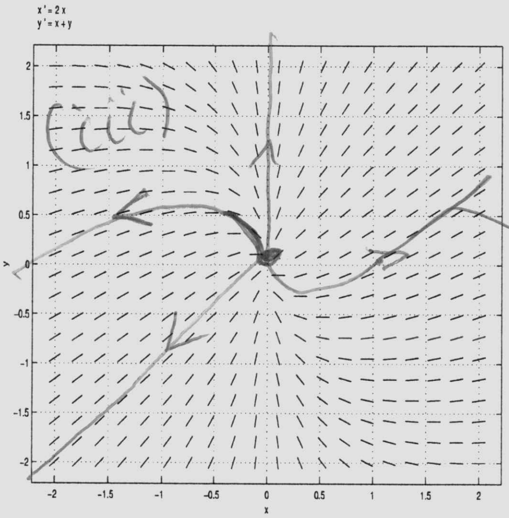
(c) The picture below shows the slope field for the system of equations. On this picture:

i. show all equilibrium solutions;

ii. draw the solution you found in part (b) above;

iii. sketch three other solution curves.

For each solution curve you draw, you should draw an arrow indicating the direction moved as time increases.



(i) • - eq sol'n

(ii)

(blank page for your working)

(a) $\det\begin{pmatrix} 2-\lambda & 0 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) = 0$

$\Rightarrow \lambda = 1, 2$

e-vector $\lambda = 1$ $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\vec{u} = 0 \Rightarrow \vec{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda = 2$ $\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}\vec{u} = 0 \Rightarrow \vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t}$

(b) $X(0) = c_1\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow c_1 = -1, \quad c_2 = 1$

$(x(t), y(t)) = (e^{2t}, -e^{t} + e^{2t})$