Department of Mathematics

MATHS 260 2008F
Mid-semester Test Answers

1. (8 marks)

   (a) Substitute $y_1(t)$ into each side of the DE:
   LHS: $dy_1/dt = 3t^{-4}$
   RHS: $3t^2y_1^2 = 3t^2(-1/t^3)^2 = 3t^2/t^6 = 3/t^4 = 3t^{-4}$.
   Since the LHS is equal to the RHS, $y_1(t)$ is a solution to the DE. 2

   (b) The DE is separable. If $y \neq 0$ we can divide through by y^2 to separate the variables:

   $$\begin{aligned} \int \frac{dy}{y^2} &= \int 3t^2\,dt \\ \Rightarrow \quad -\frac{1}{y} + c_1 &= t^3 + c_2 \\ \Rightarrow \quad y(t) &= \frac{-1}{t^3+c} \end{aligned}$$

   where c_1 and c_2 are arbitrary constants and $c = c_2 - c_1$. 4

   (c) $y(0) = 1 \Rightarrow -1/(0^3+c) = 1 \Rightarrow c = -1$, so the solution to the IVP is

   $$y(t) = \frac{-1}{t^3-1}.$$ 2

2. (16 marks)

   (a) The equilibria satisfy $y^2+2y=0$, i.e., $y=0$ and $y=-2$. Writing $f_\mu(y) = \mu + 2y + y^2$,

   $$\left.\frac{\partial f_\mu}{\partial y}\right|_{y=0} = 2 > 0$$

   so $y=0$ is a source, while

   $$\left.\frac{\partial f_\mu}{\partial y}\right|_{y=-2} = -2 < 0$$ 2

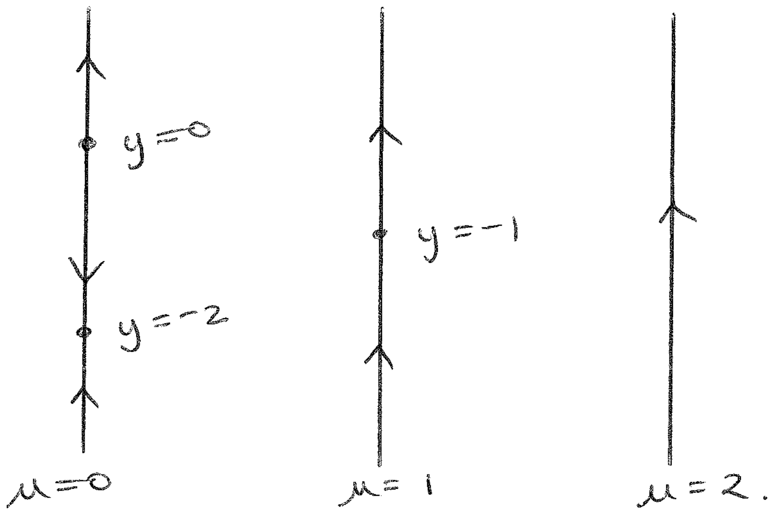
   so $y=-2$ is a sink. The phase line is as shown below.

   (b) Equilibrium solutions satisfy $y^2+2y+1 = (y+1)^2 = 0$, i.e., $y=-1$. Now

   $$\left.\frac{\partial f_\mu}{\partial y}\right|_{y=-1} = 0$$

and so linearisation is unhelpful. However, $dy/dt = (y+1)^2 \geq 0$ so all solutions except the equilibrium at $y = -1$ increase with time. Thus, the phase portrait is as shown below from which we conclude that the equilibrium solution is a node. 2

(c) There are no real equilibria in this case since $2 + 2y + y^2 = 0$ has no real solutions. Since $2+2y+y^2 = 1+(y+1)^2 > 0$, all solutions increase with time and so the phase portrait is as shown below. 1



3

(d) $\mu + 2y + y^2 = 0 \Rightarrow y = -1 \pm \sqrt{1-\mu}$, from which it is seen that there are two equilibrium solutions ($y(t) = -1 + \sqrt{1-\mu}$ and $y(t) = -1 + -\sqrt{1-\mu}$) for $\mu < 1$, one equilibrium ($y(t) = -1$) if $\mu = 1$, and no equilibria if $\mu > 1$. Now 2

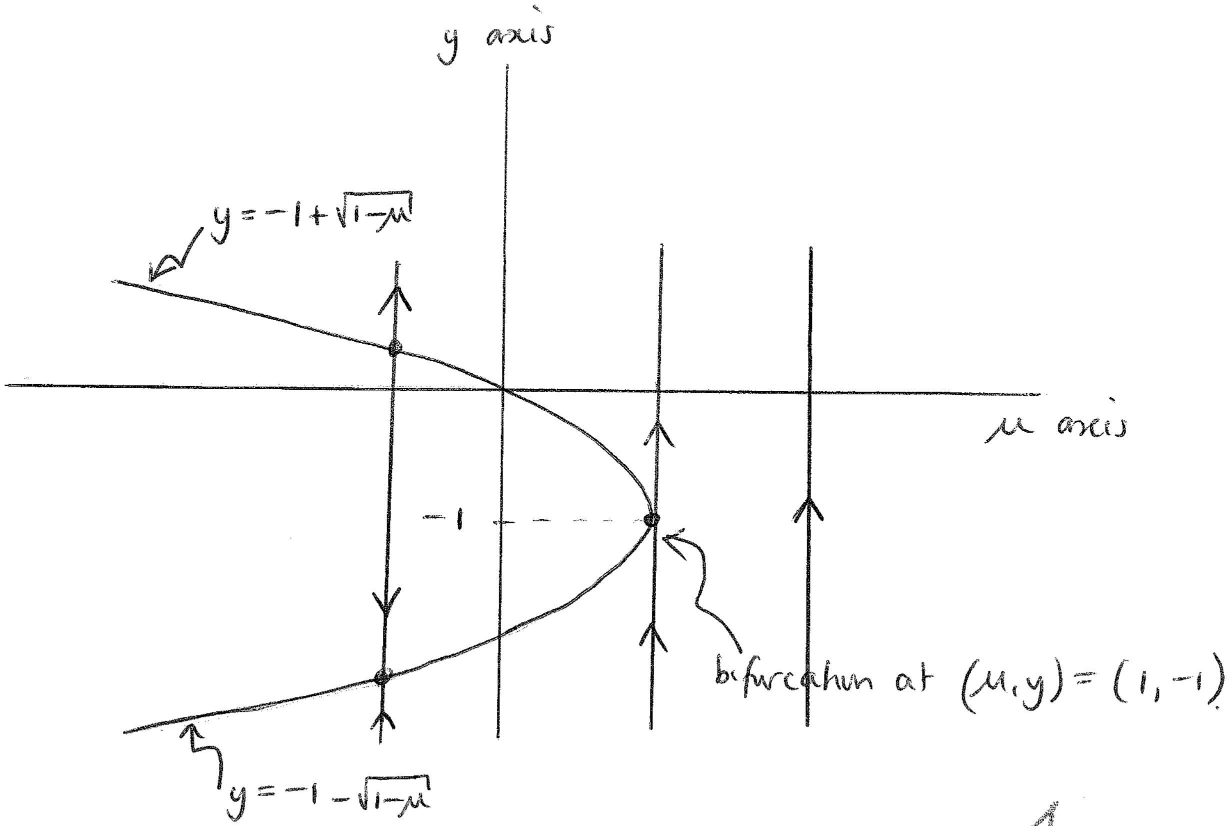
$$\left.\frac{\partial f_\mu}{\partial y}\right|_{y=-1+\sqrt{1-\mu}} = 2\sqrt{1-\mu} > 0$$ 1

if $\mu < 1$ and so $y = -1 + \sqrt{1-\mu}$ is a source for $\mu < 1$. Similarly,

$$\left.\frac{\partial f_\mu}{\partial y}\right|_{y=-1-\sqrt{1-\mu}} = -2\sqrt{1-\mu} < 0$$ 1

for $\mu < 1$ and so $y = -1 + \sqrt{1-\mu}$ is a sink for $\mu < 1$. Neither equilibrium is defined for $\mu > 1$. The equilibria coincide at $y = -1$ if $\mu = 1$ and this equilibrium is a node by (b).

There is a bifurcation at $\mu = 1$, The bifurcation diagram is shown on the next page.



4

3. (8 marks)

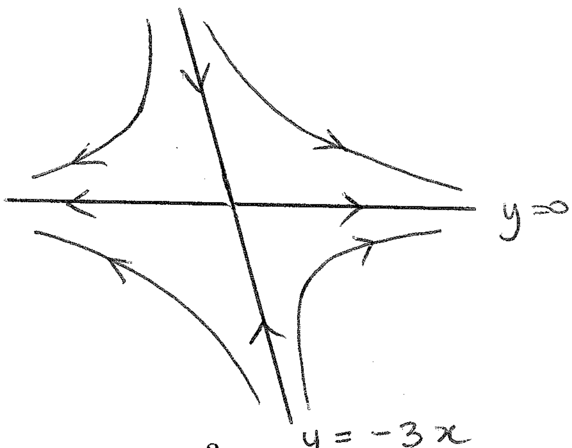
   (a) The equilibrium at the origin is a saddle if the eigenvalues of the matrix are real and of opposite sign to each other.The eigenvalues of the matrix are in fact a and -2, and so the origin is a saddle if $a > 0$. 1

   (b) When $a = 1$ the matrix has eigenvalues 1 and -2 with eigenvectors $(1, 0)^T$ and $(1, -3)^T$, and so the general solution is

$$c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

5

The phase portrait is shown below.



2.

4. (7 marks)

(a) The formulae for Euler's method are $t_{n+1} = t_n + h$ and $y_{n+1} = y_n + hf(t_n, y_n)$ where f is defined by the DE, i.e., $dy/dt = f(t, y)$. In this example, $f(t, y) = ty + t + y$, $y_0 = 0.5$, $t_0 = 1$, final $t = 3$ and so $h = 1$.

Hence $y(2) = y_1 = y_0 + hf(t_0, y_0) = 0.5 + 1 \times (0.5 \times 1 + 1 + 0.5) = 2.5$ and $y(3) = y_2 = y_1 + hf(t_1, y_1) = 2.5 + 1 \times (2.5 \times 2 + 2 + 2.5) = 12$ and so $y(3) \approx 12$. 3

(b) Two ways to get a more accurate numerical solution would be use a smaller stepsize or to use a method such as Improved Euler, that evaluates the slope at the beginning and end of the step rather than just the beginning of the step. 2

(c) To check whether a particular numerical is accurate enough, you can recompute the numerical solution using a smaller stepsize. The difference between the solutions computed with the two different stepsizes is a guide to the accuracy of the solution - if the difference is about the same size as the desired accuracy then the numerical method used is probably accurate enough. You could also recompute the solution with a much more accurate method and compare the results. 2

5. (6 marks) Analytic, qualitative and numerical methods all need to be considered.

The DE is separable, and so in principle an analytic solution could be found by separating variables, and integrating. However, the integration would not be straightforward and so this is probably not a feasible method.

Numerical methods would be useful, i.e., use a package such as dfield from Matlab to find the behaviour of solutions. For each particular choice of the constants k and E, solutions could be plotted and then equilibria determined and a phase line constructed. By repeating this process for different values of E, say, with k fixed, it would be possible to construct a bifurcation diagram. This could be repeated for different choices of k. 6