Maths 260 Lecture 16

- **Topics for today:**
  Equilibrium solutions
  Solutions for some special systems
- **Reading for this lecture:** BDH Section 2.3
- **Suggested exercises:**
  BDH Section 2.3, #1,3,5,7,9
- **Reading for next lecture:** BDH Section 3.1, 3.2
- **Today's handouts**: Tutorial 6

Equilibrium solutions

- The point $\mathbf{Y}_0$ is an **equilibrium point** for the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{V}(\mathbf{Y})$$

  if $\mathbf{V}(\mathbf{Y}_0) = 0$.

- If $\mathbf{Y}_0$ is an equilibrium point, then the constant function $\mathbf{Y}(t) = \mathbf{Y}_0$ is a solution of the system.

Example 1: Find all equilibrium solutions for the system

$$\dot{x} = 2x + y$$
$$\dot{y} = 2y + x$$

Example 2: Find all equilibrium solutions for the system

$$\dot{x} = x + y$$
$$\dot{y} = y(2 - x)$$

Example 3: Find all equilibrium solutions for the system

$$\dot{x} = 10(y - x)$$

$$\dot{y} = 28x - y - xz$$

$$\dot{z} = -\frac{8}{3}z + xy$$

Solutions near equilibria

- The behaviour of solutions near equilibria can be observed with *pplane.*
- Note that:
  1. the direction of vectors in the direction field changes dramatically near an equilibrium point, and
  2. solutions passing near an equilibrium point go very slowly, because all components of the vector field get close to zero near an equilibrium.

Analytic methods for some special systems

- Some very special systems of DEs **decouple**, i.e., the rate of change of one or more of the dependent variables depends only on its own value.

Example 4: (completely decoupled)

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= -2y \end{aligned}$$

Example 5: (partially decoupled)

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= 2x - y \end{aligned}$$

- We can sometimes find analytic solutions to systems that decouple.

Example 4 again: Find and plot solutions to the system

$$\dot{x} = x$$
$$\dot{y} = -2y$$

- We can solve each equation separately, and find that

  $$(x(t), y(t)) =$$

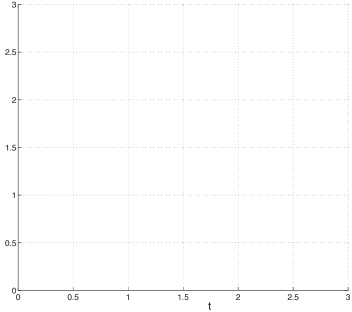
  is a solution for all choices of c_1 and c_2.
- The values of c_1 and c_2 are determined by the initial conditions:

  $$c_1 = x(0) \quad \text{and} \quad c_2 = y(0)$$

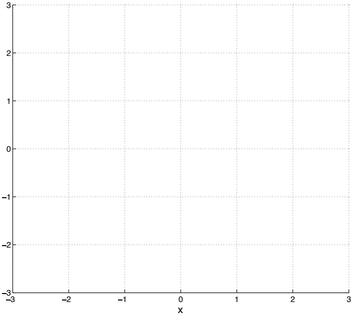
Plotting solutions

We plot the solution that satisfies $x(0) = 1, \quad y(0) = 1.$

As a function of t:



In phase space:



Solutions using other initial conditions:

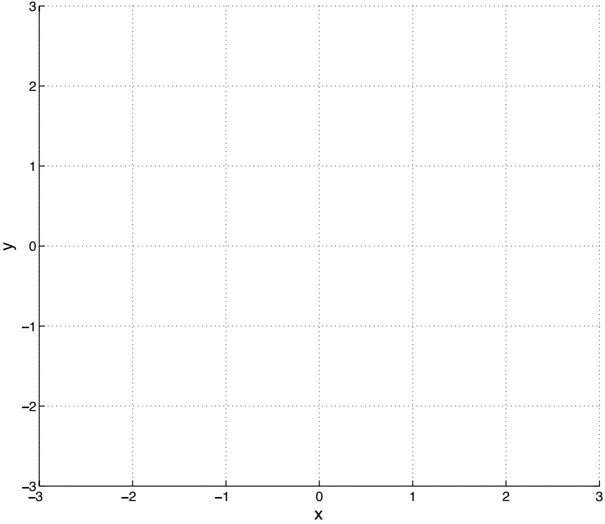
- $x(0) = 1, \quad y(0) = -1$

- $x(0) = 0, \quad y(0) = 1$

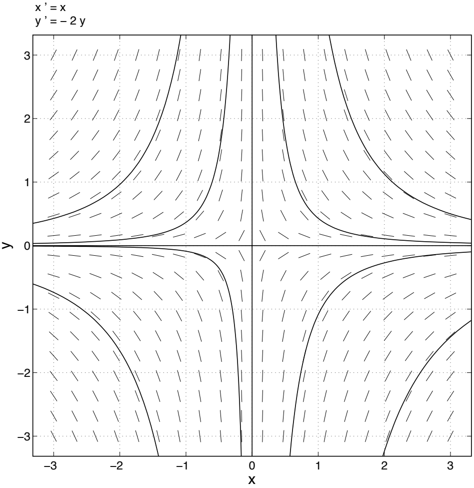
- $x(0) = -2, \quad y(0) = 1$

- $x(0) = 1, \quad y(0) = 0$

Phase space



We can compare the phase plane plot above with numerical solutions from *pplane*:



Example 5 again: Find and plot solutions to

$$\dot{x} = x$$
$$\dot{y} = 2x - y$$

- Method: solve the first equation to get $x(t)$, then substitute this expression into the second equation. This will give a linear DE for y which can be solved.

- We find
$$(x(t), y(t)) = (c_1 e^t, c_1 e^t + c_2 e^{-t})$$
is a solution for all choices of the constants c_1 and c_2.
- We can also write this in vector form:
$$\mathbf{Y}(t) = \begin{pmatrix} c_1 e^t \\ c_1 e^t + c_2 e^{-t} \end{pmatrix} = c_1 \mathbf{Y_1} + c_2 \mathbf{Y_2},$$
where $\mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$ and
$$\mathbf{Y}_1(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}, \quad \mathbf{Y}_2(t) = \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix}$$

Plotting solutions in phase space

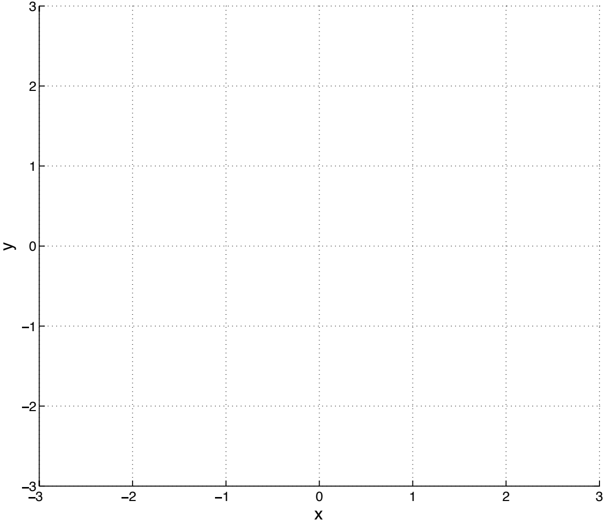
- $x(0) = 1, \quad y(0) = 1$

- $x(0) = 0, \quad y(0) = 1$

- $x(0) = 1, \quad y(0) = 0$

- $x(0) = 0.25, \quad y(0) = 1$

Phase space



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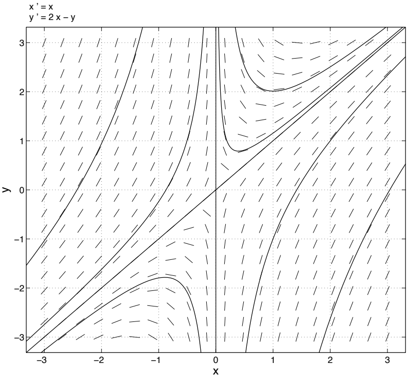

In both examples, an analytic solution to the system could be found.

- We found that some solutions gave straight-line solution curves in phase portrait, but
- (from *pplane*) most solution curves are not straight lines.

Important ideas from today's lecture:

- The point $\mathbf{Y}_0$ is an **equilibrium point** of the DE system if $\mathbf{V}(\mathbf{Y}_0) = 0$, in which case the constant function $\mathbf{Y}(t) = \mathbf{Y}_0$ is a solution of the system.

- We can find analytic solutions for systems that decouple.