Maths 260 Lecture 33

- **Topics for today:**
  More nonhomogeneous higher order DEs
  Forcing and resonance in the harmonic oscillator
- **Reading for this lecture:** BDH Section 4.3, 4.4
- **Suggested exercises:** BDH Section 4.3; 3,7,9, Section 4.4; 2
- **Reading for next lecture:** None
- **Today's handouts:** Tutorial 12

Recap: Method of solution for equations of the form

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = f(t)$$

- Find the general solution to the related homogeneous equation.
- Find *one* solution to the nonhomogeneous equation.
- Add answers to steps 1 and 2 to get the general solution to the nonhomogeneous equation.
- If trying to solve an IVP, use the initial conditions to determine constants in the general solution.

To find a particular solution to the DE

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = f(t)$$

where $f(t)$ is

- constant, or
- t^n for n a positive integer, or
- $\mathrm{e}^{\lambda t}$ for real nonzero λ, or
- $\sin(bt)$ or $\cos(bt)$, for b constant, or
- a finite product of terms like these

Step 1: Form the UC set consisting of f and all linearly independent functions obtained by repeated differentiation of f.

Step 2: If any of the functions in the UC set is also a solution to the homogeneous DE, multiply all functions in the set by t^k, where k is the smallest integer so that the modified UC set does not contain any solutions to the homogeneous DE.

Step 3: Find a particular solution to the DE by taking a linear combination of all the functions in the (possibly modified) UC set. Determine the unknown constants by substituting this linear combination into the DE.

We can modify this method to solve DEs for which the forcing function is a finite sum of terms.

For example, to find a solution to

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = f_1(t) + f_2(t)$$

we use linearity, i.e., first find y_1 that solves

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = f_1(t)$$

and then find y_2 that solves

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = f_2(t)$$

Then $y = y_1 + y_2$ solves the DE with forcing term $f(t) = f_1(t) + f_2(t)$.

Example 1:

Find the general solution to

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 2\sin t - \mathrm{e}^{2t}$$

Find a solution to $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 2\sin t$

Then find a solution to $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = -\mathrm{e}^{2t}$

Example 2:

Find the general solution to

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} + 5\frac{dy}{dt} = t + 2\mathrm{e}^t$$

Find a solution to $\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} + 5\frac{dy}{dt} = t$

Then find a solution to $\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} + 5\frac{dy}{dt} = 2\mathrm{e}^t$

The forced harmonic oscillator

A higher order equation of special interest in applications is the periodically forced harmonic oscillator, i.e.

$$\frac{dy}{dt} + p\frac{dy}{dt} + qy = \cos(\omega t)$$

for constants $p \geq 0$, $q > 0$ and $\omega > 0$.

We are interested in the long term behaviour of solutions for various values of the damping coefficient p.

The characteristic polynomial is $\lambda^2 + p\lambda + q = 0$, which has roots

$$\lambda = -\frac{p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2}$$

Thus, in the case $0 \leq p^2 < 4q$, the homogeneous equation has general solution

$$y_c = c_1 \mathrm{e}^{-\frac{p}{2}t} \cos(\alpha t) + c_2 \mathrm{e}^{-\frac{p}{2}t} \sin(\alpha t)$$

where

$$\alpha = \frac{\sqrt{4q - p^2}}{2}$$

Exercise: Find the general solution to the homogeneous problem when $p^2 \geq 4q$.

Exercise: Show that for all values of $p > 0$ the solution to the homogeneous equation tends to zero as $t \to \infty$.

A consequence of the result in the previous exercise is that as $t \to \infty$, all solutions to the forced harmonic oscillator with nonzero damping behave the same, i.e. like the particular solution to the non-homogeneous problem.

To find the particular solution:

We find

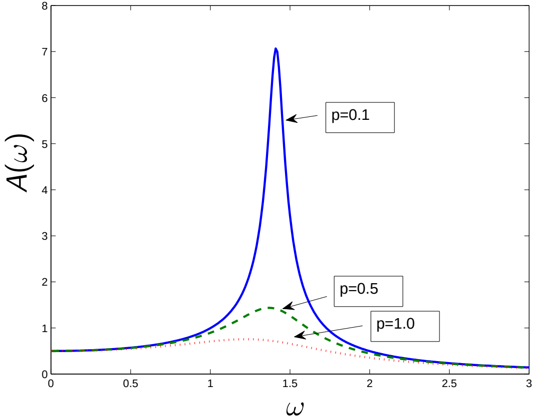
$$y_p = A\cos(\omega t + \theta)$$

where

$$A = \frac{1}{\sqrt{(q-\omega^2)^2 + (\omega p)^2}}, \quad \theta = \tan^{-1}\left(\frac{-\omega p}{q-\omega^2}\right)$$

Qualitative behaviour of this solution

Amplitude of y_p as a function of ω in the case $q = 2$ and for various p:



In the undamped case ($p = 0$), a particular solution is

$$y_p = \begin{cases} \dfrac{1}{q - \omega^2} \cos \omega t, & \omega^2 \neq q \\ \\ \dfrac{1}{2\omega} t \sin \omega t, & \omega^2 = q \end{cases}$$

Exercise: Check this.

Qualitative behaviour of this solution:

Summary:

For the damped periodically forced harmonic oscillator

$$\frac{dy}{dt} + p\frac{dy}{dt} + qy = \cos(\omega t)$$

all solutions eventually become periodic with frequency the same as the forcing frequency, ω, and with amplitude depending on ω.

The amplitude of the long term solutions can be very large if the forcing frequency, ω, is close to the natural frequency of the unforced system ($\sqrt{q}$).

Also, smaller damping (p) leads to a larger amplitude for the long term solutions.

Important ideas from today

To find a solution to

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = f_1(t) + f_2(t)$$

we use linearity, i.e. first find y_1 that solves

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = f_1(t)$$

and then find y_2 that solves

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = f_2(t)$$

Then $y = y_1 + y_2$ solves the DE with forcing term $f(t) = f_1(t) + f_2(t)$.

For the damped periodically forced harmonic oscillator

$$\frac{dy}{dt} + p\frac{dy}{dt} + qy = \cos(\omega t)$$

all solutions eventually become periodic:

- The frequency of the periodic behaviour is the same as the forcing frequency, ω,
- The amplitude of the periodic behaviour depends on ω, i.e. the amplitude is larger if the forcing frequency, ω, is closer to $\sqrt{q}$, the natural frequency of the unforced system.