Maths 260 Lecture 25

- **Topics for today:**
  - Non-linear systems: linearisation near equilibria
  - Classification of equilibria in nonlinear systems
- **Reading for this lecture:** BDH Section 5.1
- **Suggested exercises:** BDH Section 5.1; 1, 3, 7, 9, 11
- **Reading for next lecture:** BDH Section 5.2
- **Today's handouts:** None

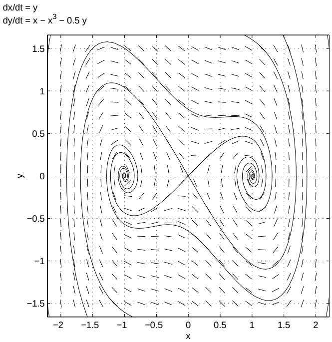
Example 1:

Consider the system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = x - x^3 - \frac{1}{2}y$$

Equilibrium solutions:

Slope field and some solutions:



The equilibrium at the origin looks like a saddle in a linear system, and the other equilibria look like spiral sinks.

We can understand the saddle-like nature of $(0,0)$ if we approximate the nonlinear system by a linear system.

For x, y very close to zero, x^3 is much smaller than x or y. So we can ignore the x^3 term in the nonlinear system, and approximate the nonlinear system near $(0,0)$ by the linear system

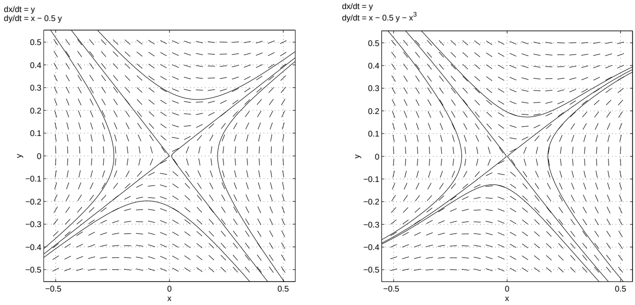
$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = x - \frac{1}{2}y$$

i.e.

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY} = \begin{pmatrix} 0 & 1 \\ 1 & -\frac{1}{2} \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues of matrix **A** are 0.78 and -1.28, so the equilibrium at the origin of the linear system is a saddle.

The following pictures show the slope field and solutions for the linear system (on left) and an enlargement of the nonlinear system near the origin (on right).

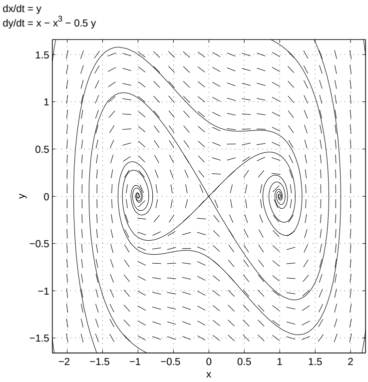


Note that the linear system is a good approximation to the nonlinear system near the equilibrium, but is hopeless away from the equilibrium (see the earlier phase portrait).

The procedure used above is called linearisation:

- Near an equilibrium, approximate the nonlinear system by an appropriate linear system.
- For initial conditions near the equilibrium, solutions of the nonlinear system stay close to solutions of the approximate linear system, at least for some interval of time.
- Thus, the type of equilibrium at the origin in the linearised system gives information about the type of the corresponding equilibrium in the nonlinear system.

Returning to the original system in Example 1, we now consider the equilibria at $(1, 0)$ and $(-1, 0)$.



- To approximate the behaviour near $(1, 0)$ by a linear system, we must first shift the equilibrium to the origin - because linear systems usually only have an equilibrium at the origin.
- We change the coordinates as follows:
  Write $u = x - 1$, $v = y$, so the equilibrium $(x, y) = (1, 0)$ is now at $(u, v) = (0, 0)$.
- Then the system becomes:

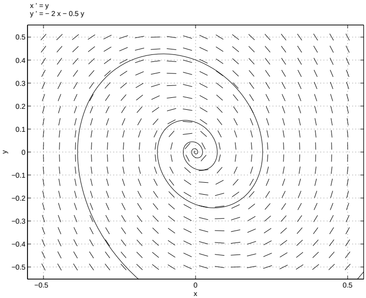
$$\frac{du}{dt} =$$
$$\frac{dv}{dt} =$$

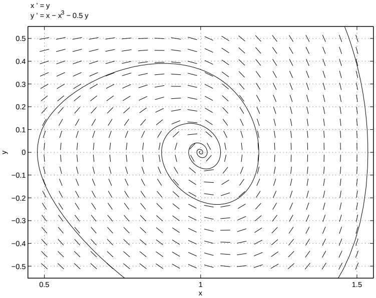
For u and v small, $-3u^2$ and u^3 are very, very small. We ignore these nonlinear terms and approximate the system by:

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Eigenvalues are $-\dfrac{1}{4} \pm \dfrac{1}{4}\sqrt{31}i$. So the origin is a spiral sink in the linear approximation.

The following pictures illustrate the similarity between the phase portrait for the linearised system (on the left) and the phase portrait near the equilibrium at $(1, 0)$ in the nonlinear system (on the right).





Similar calculations give similar results for the equilibrium at $(-1, 0)$.

More generally...

If the system

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

has an equilibrium at (x_0, y_0), we can construct a linear approximation to the system, valid for x and y values near (x_0, y_0), as follows.

- First move the equilibrium to the origin: write $u = x - x_0$ and $v = y - y_0$. The nonlinear equations in the new coordinates are:

$$\frac{du}{dt} = \frac{dx}{dt} = f(x,y) = f(x_0 + u, y_0 + v)$$
$$\frac{dv}{dt} = \frac{dy}{dt} = g(x,y) = g(x_0 + u, y_0 + v)$$

- Now we use a Taylor expansion to rewrite f and g, because u and v are small:

$$f(x_0+u, y_0+v) = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right] u + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right] v + \text{h.o.t}$$

$$g(x_0+u, y_0+v) = g(x_0, y_0) + \left[\frac{\partial g}{\partial x}(x_0, y_0)\right] u + \left[\frac{\partial g}{\partial y}(x_0, y_0)\right] v + \text{h.o.t}$$

- Recall that $f(x_0, y_0) = g(x_0, y_0) = 0$, so if we ignore the higher order terms then we get an approximate linear system:

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Thus, the behaviour of solutions to the nonlinear system near the equilibrium (x_0, y_0) can be approximated by the behaviour of solutions in the linearised system given above.

The matrix of partial derivatives

$$\begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

is called the Jacobian matrix, evaluated at (x_0, y_0).

Example 1 again

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = x - x^3 - \frac{1}{2}y$$

$$J(x, y) =$$

For an equilibrium solution in a nonlinear system:

- The equilibrium is a (nonlinear) sink if all solutions that start close to the equilibrium stay close to the equilibrium for all time and tend to the equilibrium as t increases.
- The equilibrium is a (nonlinear) source if all solutions that start close to the equilibrium move away from the equilibrium as t increases.
- The equilibrium is a (nonlinear) saddle if there are curves of solutions that tend towards the equilibrium as t increases and curves of solutions that tend towards the equilibrium solution as t decreases. All other solutions started near the equilibrium move away from the equilibrium as t increases and decreases.

These are different definitions than those used for equilibria in linear systems, but are consistent with those.

Example 2

Consider the system

$$\frac{dx}{dt} = x(1 + x^2)$$
$$\frac{dy}{dt} = 3y(1 - y - x)$$

Find the equilibria and determine their types. For each equilibrium, sketch a phase portrait showing the behaviour of solutions in the associated linearised system.

$$J(x,y) = \begin{pmatrix} 1+3x^2 & 0 \\ -3y & 3-6y-3x \end{pmatrix}$$

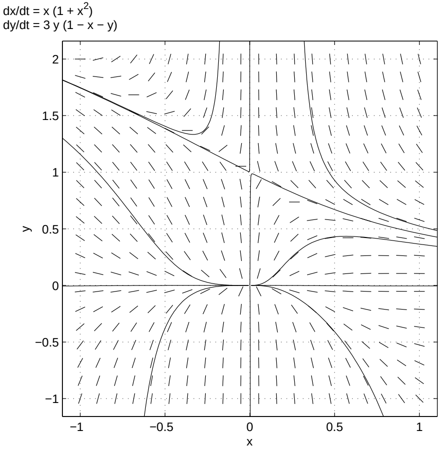
$$J(0,0) =$$

$$J(0,1) =$$

At $(0,0)$, associated linear system is $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{Y}$.

At $(0,1)$, associated linear system is $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 0 \\ -3 & -3 \end{pmatrix} \mathbf{Y}$.

The phase portrait for this system, drawn with `pplane`, is given below. Note the source at $(0, 0)$ and the saddle at $(0, 1)$ as predicted by our calculations.



Example 3

Consider the system

$$\frac{dx}{dt} = -x + y$$
$$\frac{dy}{dt} = x - y^2$$

Find the equilibria and determine their types. For each equilibrium, sketch a phase portrait showing the behaviour of solutions in the associated linearised system.

At $(0,0)$, associated linear system is $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{Y}$.

At $(1,1)$, associated linear system is $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{Y}$.

Important ideas from today

If a system of nonlinear equations

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

has an equilibrium at (x_0, y_0), then the behaviour of solutions near that equilibrium can be approximated by the behaviour of solutions near the origin for the linearised system

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

We make this idea more rigorous in the next lecture.