Maths 260 Lecture 2

- **Topics for today:**
  Solutions to differential equations
  Separable differential equations
- **Reading for this lecture:** BDH Section 1.2
- **Suggested exercises:**
  BDH Section 1.2: 1, 3, 7, 15, 25
  Revise integration by parts
- **Reading for next lecture:** BDH Section 1.3
- **Today's handouts:**
  An introduction to software used in this course
  Installing the software needed for Maths 260
  Tutorial 1 questions

Solutions of differential equations

We are interested in first order differential equations (or DEs).
A DE is said to be in **standard form** when it is written as:

$$\frac{dy}{dt} = f(t, y)$$

where:

- t is the **independent variable**, i.e., it does not depend on any other variable.
- y is the **dependent variable**, i.e, it is a function of the independent variable t. We write $y(t)$.
- f is a function of two variables, t and y.

A **solution** of a DE is any function $\phi(t)$, that, when substituted for y in the DE, satisfies the DE for all t.
That is,

$$\frac{d\phi}{dt} = f(t, \phi(t)) \qquad \text{for all } t$$

Example 1: Work out which of the following functions is a solution to the differential equation

$$\frac{dy}{dt} = \frac{y^2 - 1}{t^2 + 2t}$$

(a) $\phi_1(t) = t + 1$
(b) $\phi_2(t) = 2t + 1$
(c) $\phi_3(t) = 1$

A solution like ϕ_3 in the last example, i.e., a solution of the differential equation that does not vary with time, is called an **equilibrium solution**,

Warning: A given solution to a DE might not exist for all time.

For example, the function $\phi_1(t) = t + 1$ is a solution to the DE in Example 1 and is defined for all t.

Similarly $\phi_3(t) = 1$ is a solution to the same DE that exists for all time.

However the function

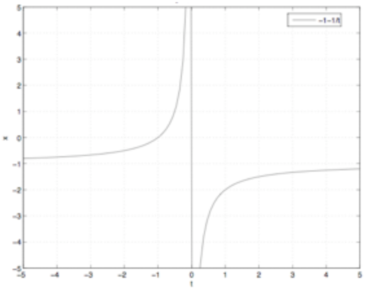
$$\phi_4(t) = -1 - \frac{1}{t}$$

is a solution to the DE

$$\frac{dy}{dt} = (1 + y)^2$$

but is not defined for $t = 0$.

Plotting the function ϕ_4 yields the following picture.



We see that ϕ_4 is really two solutions, one defined for $t > 0$ and one defined for $t < 0$.

In general, we require solutions to be continuous and say that discontinuous functions like ϕ_4 represent more than one solution, one solution for each continuous piece of the graph of the function.

Initial Value Problems

An **initial condition** tells us the value of a solution to the DE at a particular value of the independent variable.

A DE with an initial condition is called an **initial value problem.**

Example 2: Consider the initial value problem (IVP)

$$\frac{dy}{dt} = te^{t^2} + 3, \qquad y(0) = -1.$$

The function

$$\phi_g(t) = \frac{1}{2}e^{t^2} + 3t + c$$

is a solution to the differential equation for all values of c, but only the choice $c = -3/2$ satisifies the initial condition $y(0) = -1$.

Separable Equations

It is usually not possible to find actual functions that solve a particular DE, but there are a few special cases when we can calculate explicit solutions. Separable equations are one such case.

A differential equation

$$\frac{dy}{dt} = f(t, y)$$

is called **separable** if the function f can be written as

$$f(t, y) = g(t)h(y)$$

for some functions g and h.

Special cases:

- $\dfrac{dy}{dt} = g(t), \qquad \text{(i.e., } h(y) = 1).$

- $\dfrac{dy}{dt} = h(y), \qquad \text{(i.e., } g(t) = 1).$

The second example is called an **autonomous** equation because the right hand side does not depend explicitly on t.

Solving Separable Equations

A separable DE can be written as

$$\frac{dy}{dt} = g(t)h(y)$$

for some functions g and h.

If $h(y) \neq 0$, divide by $h(y)$:

$$\frac{1}{h(y)}\frac{dy}{dt} = g(t), \qquad \text{i.e.} \quad \frac{1}{h(y(t))}\frac{dy}{dt} = g(t)$$

Integrate with respect to t, if possible:

$$\int \frac{1}{h(y(t))}\frac{dy}{dt}dt = \int g(t)dt$$

By the chain rule,

$$dy = \frac{dy}{dt}dt$$

so

$$\int \frac{1}{h(y)}dy = \int g(t)dt$$

If we can do these integrals we can get an expression for $y(t)$, the solution to the DE.

Example 3: $\dfrac{dy}{dt} = t^3 y$

Note that $y(t) = 0$ is a solution to this DE (check this!) but is not found by this method. We call such a solution a **missing solution**).

Example 4: $\dfrac{dy}{dt} = \dfrac{t}{1+y^2}$

Example 5: $\dfrac{dy}{dt} = e^{t^2}$

Example 6: $\dfrac{dy}{dt} = \dfrac{te^{-t}}{y},\ y(0) = 2$

Important ideas from today:

- It is always possible to check whether a proposed solution to a DE really is a solution by substituting it into the DE.
- We learnt about independent and dependent variables; equilibrium solutions; and initial value problems.
- We learnt a technique for solving separable equations and how to check for missing solutions.