Maths 260 Lecture 21

- **Topics for today:**
  Complex numbers:
  - De Moivre's formula
  - Derivatives of complex-valued functions
  - Euler's formula
  - The exponential of a complex number
- **Reading for this lecture:**
  Some notes on complex numbers
  BDH Appendix C
- **Suggested exercises:**
  Problems at the back of "Some notes on complex numbers"

Multiplication using polar form

- Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

  be any two complex numbers. Then

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

- Proof:

- Hence, **multiplying** complex number in polar form corresponds to taking the product of the moduli and the sum of the arguments.

Example 1

- Evaluate $2i(1+i)$ using both polar and rectangular forms. Compare your answers.

Example 2: Calculate $(\cos\theta + i\sin\theta)^2$

Example 3: Calculate $(\cos\theta + i\sin\theta)^3$

de Moivre's formula

- Examples 2 and 3 give particular cases of **de Moivre's formula:**

$$(\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta).$$

Example 4: Use de Moivre's formula to express $\cos 2\theta, \sin 2\theta$ in terms of $\cos\theta, \sin\theta$.

Polar form and solving equations

- **Example 5:** Find all solutions to the equation $z^3 = 1$.
- $z = 1$ is obviously a solution. Any others? We expect three solutions from the Fundamental Theorem of Algebra.
- Write
$$z = r(\cos\theta + i\sin\theta),$$
where $r = |z| > 0$.
- Then
$$z^3 = r^3(\cos 3\theta + i\sin 3\theta)$$
and therefore
$$r^3(\cos 3\theta + i\sin 3\theta) = 1$$
- So...

- Note that we get only three distinct solutions because of the periodicity of cosine and sine.
- Plot the solutions in the Argand plane:

Example 6

Find all solutions to the equation $z^3 = 1 + i$.

Derivatives of complex valued functions

- Suppose t is real and $f(t)$ is a complex-valued function of t, i.e.

$$f(t) = u(t) + iv(t)$$

  for some real-valued functions u and v.

- Then, if u and v are differentiable with respect to t, we define the derivative of $f(t)$ to be

$$\frac{df}{dt} = \frac{du}{dt} + i\frac{dv}{dt}$$

Example 7:

Find the derivative of the function $f(t) = \cos(t) + i\sin(t)$.

Properties of the function $f(t) = \cos(t) + i\sin(t)$:

- $f'(t) = if(t)$,
- $f(0) = 1$,
- $f(t_1)f(t_2) = f(t_1 + t_2)$.

Compare this to the function $g(t) = e^{at}$, where a is real:
Properties of $g(t)$:

- $g'(t) =$
- $g(0) =$
- $g(t_1)g(t_2) =$

Euler's formula

The similarities between the properties of f and g prompted Euler to make the definition:

Euler's Formula:

$$e^{it} = \cos t + i \sin t$$

Euler's Formula and Polar form

- **Example 8:** Rewrite $z = 1 + i$ using the complex exponential.

- In general, complex number $z = a + ib$ can be written in polar form as

$$z = re^{i\theta}$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$.

Arithmetic in exponential form

- Now **multiplication** and **division** are easy.

Example 9: If $z_1 = 2e^{i\pi/6}$ and $z_2 = -e^{i\pi/4}$, compute $z_1 z_2$ and z_1/z_2

Arithmetic in exponential form

- We can easily calculate **powers** using complex exponentials.

Example 10: If $z = 3e^{i\pi/5}$, find z^2 and z^5.

Other properties of complex exponentials

- We have: $e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$

Example 11: Calculate $e^{(2+3i)t}$ and find the real and imaginary parts

Example 12: Calculate $e^{(-1-4i)t}$ and find the real and imaginary parts

Derivatives in exponential form

- If λ is a complex number then
$$\frac{d}{dt}\left(e^{\lambda t}\right) = \lambda e^{\lambda t}.$$
- Proof:

Important ideas from today's lecture:

- de Moivre's formula

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta).$$

- Euler's formula

$$e^{it} = \cos t + i\sin t$$

- Derivatives of complex-valued functions
- The complex exponential