Department of Mathematics

Maths 260 Assignment 2 Solutions

August 25, 2009

1. **First phase line:**

(a) One possible differential equation is

$$\frac{dy}{dt} = (y+1)^2,$$ 2

but other choices are also possible.

(b) If $y_0 < -1$, $y(t) \to -1$ as $t \to \infty$. If $y_0 > -1$, $y(t) \to \infty$ as t increases. If $y_0 = -1$, then $y(t) = -1$ for all t. 2

Second phase line:

(a) One possible differential equation is

$$\frac{dy}{dt} = -y(y-2)(y-1)$$ 2

but other choices are also possible.

(b) If $y_0 < 0$, $y(t) \to 0$ as $t \to \infty$. If $0 < y_0 < 1$, $y(t) \to 0$ as $t \to \infty$. If $1 < y_0 < 2$, $y(t) \to 2$ as $t \to \infty$. If $y_0 > 2$, $y(t) \to 2$ as $t \to \infty$. If $y_0 = 0$ then $y(t) = 0$ for all t, if $y_0 = 1$ then $y(t) = 1$ for all t, and if $y_0 = 2$ then $y(t) = 2$ for all t. 3

Third phase line:

(a) One possible differential equation is

$$\frac{dy}{dt} = 1$$ 2

but other choices are also possible. 1

(b) All solutions have $y(t) \to \infty$ as t increases.

In all cases, the dfield plot will depend on the differential equation chosen in (a). 3

Q1 Total 15

2. (a) This is a linear equation. Rewrite the equation as

$$\frac{dy}{dt} - \frac{y}{t} = -1.$$

Then the integrating factor is

$$\mu(t) = \exp\left(\int -\frac{1}{t}dt\right) = \exp(-\ln|t|) = \frac{1}{t}.$$ 2

Here I have got rid of the absolute value signs because $t > 0$ and I have chosen the constant of integration to be zero.

Multiplying the DE by $\mu(t)$ yields

$$\frac{1}{t}\frac{dy}{dt} - \frac{y}{t^2} = -\frac{1}{t}$$

or

$$\frac{d}{dt}\left(\frac{y}{t}\right) = -\frac{1}{t}.$$

Integrating both sides with respect to t gives

$$\frac{y}{t} = -\ln|t| + c$$

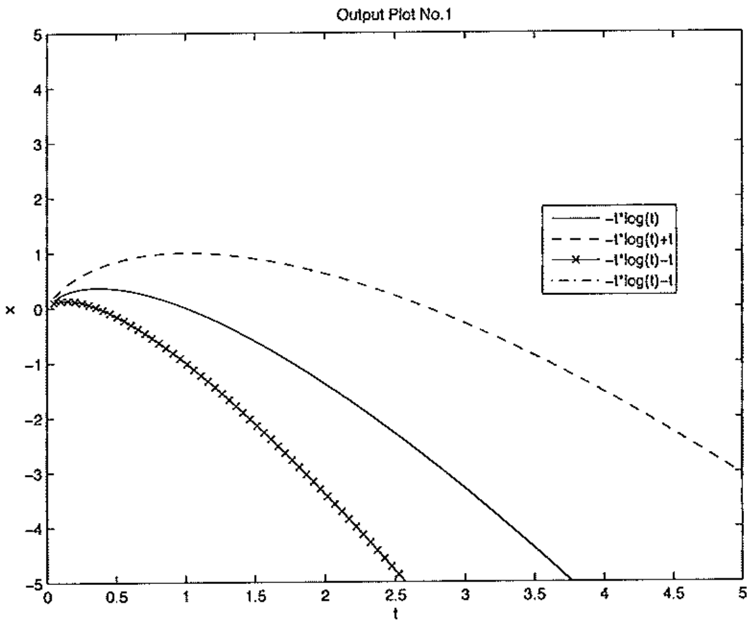
where c is an arbitrary constant. The absolute value signs can be removed because $t > 0$ and rearranging gives 1

$$y(t) = -t\ln t + ct.$$

3.

This is a one-parameter family of solutions to the DE.

(b) Here is an analyzer picture using the choices $c = 0$, $c = 1$ and $c = -1$.



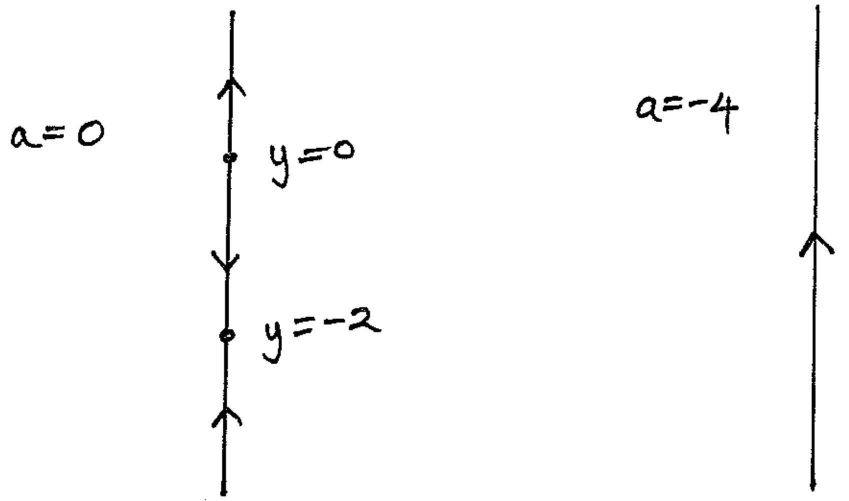
3

(c) All solutions will have $y(t) \to -\infty$ as $t \to \infty$. Even if solutions initially increase because c is large and positive, eventually for large enough t, the term $-t\ln t$ will be larger in magnitude than the term ct and so all solutions will tend to $-\infty$. 1

Q2 10

3. (a) Equilibrium solutions satisfy $y^2 + 2y = 0$, i.e., $y = 0$ and $y = -2$. Writing $f(y) = y^2 + 2y$ we see that $\partial f/\partial y = 2y + 2$. At $y = 0$, $\partial f/\partial y = 2$ so $y = 0$ is a source. At $y = -2$, $\partial f/\partial y = -2$ so $y = -2$ is a sink. The phase line is as shown on the next page. 4

(b) Equilibrium solutions satisfy $y^2 + 2y + 4 = 0$ or $y = (-2 \pm \sqrt{4 - 16})/2$. Thus there are no equilibria for this value of a since we are only interested in real values of y. Also, $y^2 + 2y + 4 = (y + 1)^2 + 3 > 0$ for all y. Hence the phase line is as shown on the next page. 2



2

(c) i. Equilibria satisfy $y^2+2y-a=0$, i.e., $y=(-2\pm\sqrt{4+4a})/2=-1\pm\sqrt{1+a}$. Thus there are two equilibria for $a>-1$, one equilibrium for $a=-1$ and no equilibria for $a<-1$. Also, writing $f(y)=y^2+2y-a$, we see that $\partial f/\partial y=2y+2$. At $y=-1+\sqrt{1+a}$ 2

$$\frac{\partial f}{\partial y}=2\sqrt{1+a}$$

so $y=-1+\sqrt{1+a}$ is a source for $a>-1$. At $y=-1-\sqrt{1+a}$, 2

$$\frac{\partial f}{\partial y}=-2\sqrt{1+a}$$

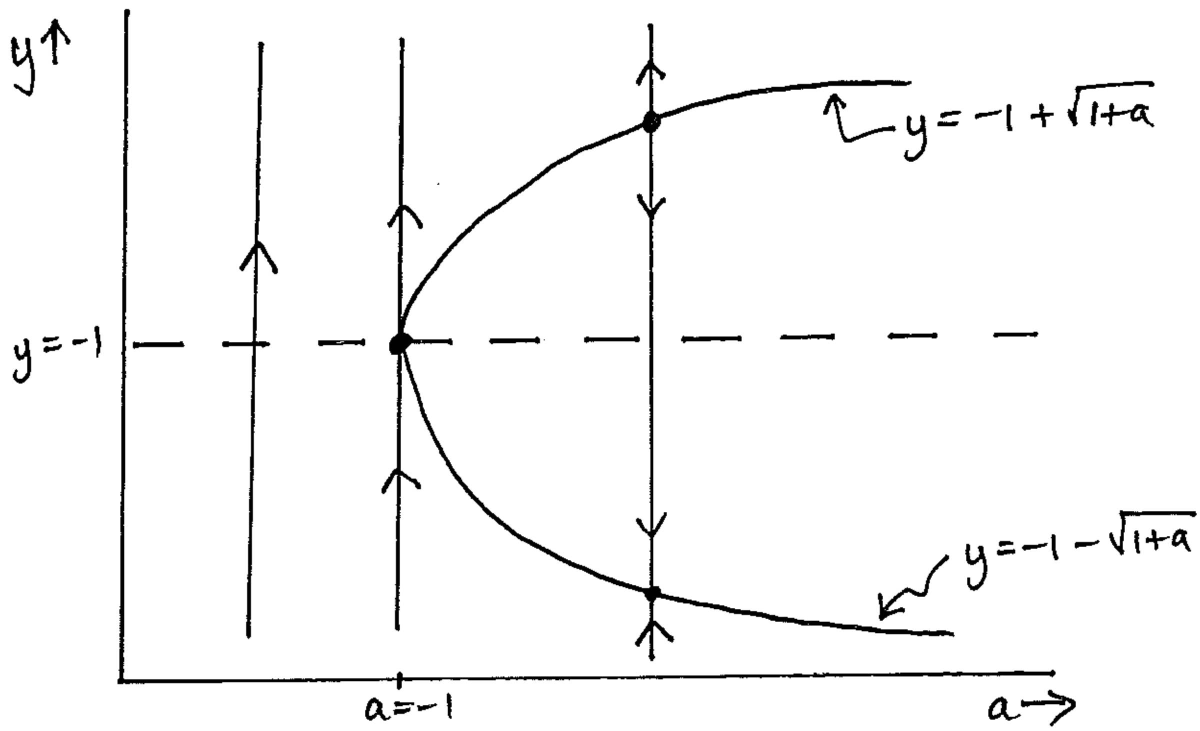
so $y=-1-\sqrt{1+a}$ is a sink for $a>-1$. 2

At $a=-1$, linearization fails, but

$$\frac{dy}{dt}=y^2+2y+1=(y+1)^2\geq 0$$

so $y=-1$ is a node. 2

ii. The bifurcation diagram is shown below.



6

Q3 Total 22

4. (a) Expanding out the bracket gives

$$\frac{dc}{dt} = 100kc - kc^2 - s.$$

The term $100kc$ represents the growth rate if $s = 0$ and if c is small (so $kc^2 << 100kc$). In this case, the population grows at a rate proportional to the current size of the population. The term $-kc^2$ becomes significant for larger c; this term ensures that the population cannot grow without bound by making the growth rate negative if the population is too large ($100kc - kc^2 < 0$ if $c > 100$). The term $-s$ removes cockroaches from the population at a constant rate; this term could represent trapping or spraying (poisoning) of the cockroaches at a constant rate. 3

(b) For $c << 100$, $100kc \approx 0.1c$, i.e., $k = 0.001$ will do. 1

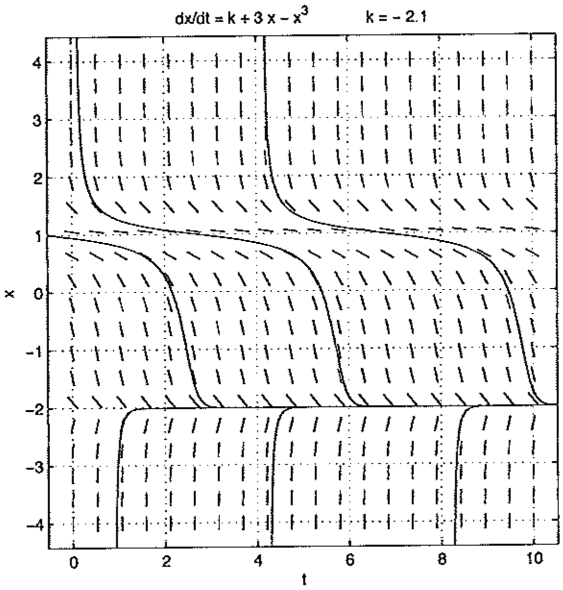
(c) Equilibrium values for the population occur when $100kc - kc^2 - s = 0$, i.e., $c = (-100k \pm \sqrt{(100k)^2 - 4ks})/(-2k) = 50 \pm \sqrt{(50k)^2 - ks/k}$, which takes on its maximum value when $s = 0$. In this case, maximum c is 100, so the maximum equilibrium population is 100,000 cockroaches. 2

Q4 6

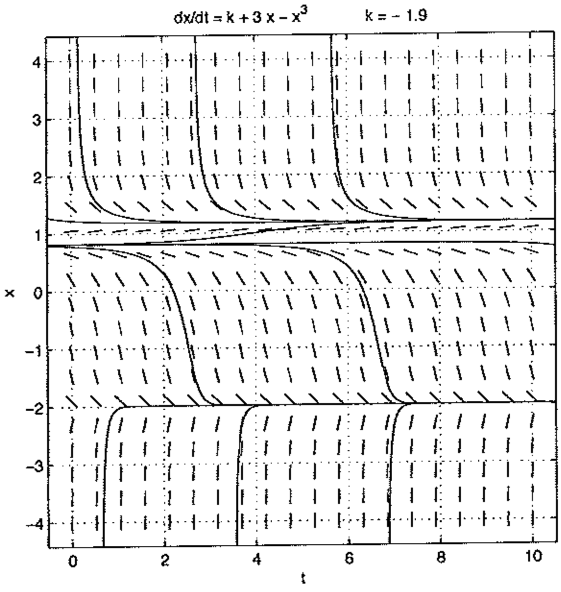
5. (a) Bifurcations occur at $k = -2$ and $k = 2$. For $k < -2$, there is one equilibrium solution (a sink, at negative y) and for $k > -2$ but $k \approx -2$ there are three equilibria, a sink at negative y and two (one source, one sink) for $y \approx 1$. Thus as k passes through -2 from below, a pair of equilibria appear (this is a saddle-node bifurcation). 4 2

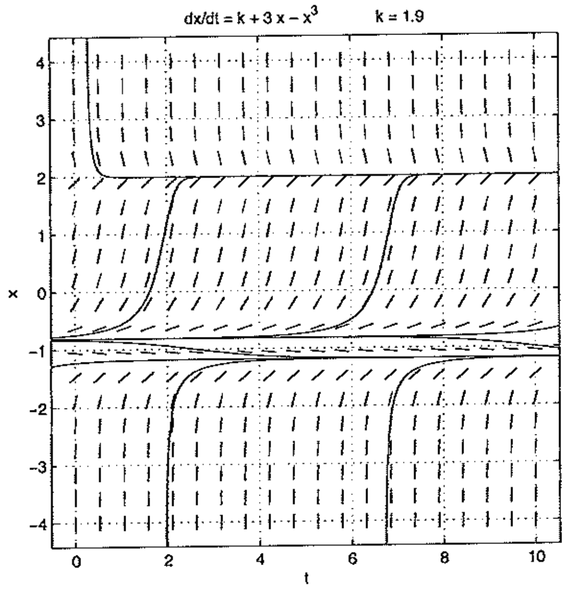
For $k > 2$, there is one equilibrium solution (a sink, at positive y) and for $k < 2$ but $k \approx 2$ there are three equilibria, a sink at positive y and two (one source, one sink) for $y \approx -1$. Thus as k passes through 2 from below, a pair of equilibria disappear (this is also a saddle-node bifurcation). 2

These observations can be confirmed by inspecting dfield plots, as shown in the figures below. It is also possible to work this out analytically, by plotting $f(y) = k + 3y - y^2$ for various choices of k, noting that the turning points of f are always at $y = \pm 1$ and therefore deducing that there are three roots of $f(y) = 0$ if $-2 < k < 2$, one root if $k = -2$ or $k = 2$ and no roots otherwise.



4 for nice pictures supporting the description


dx/dt = k + 3 x − x3
k = − 1.9
x
t


dx/dt = k + 3 x − x3
k = 1.9
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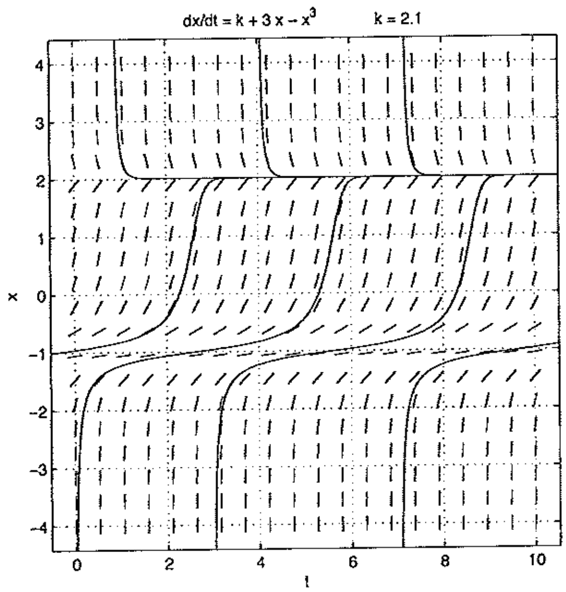
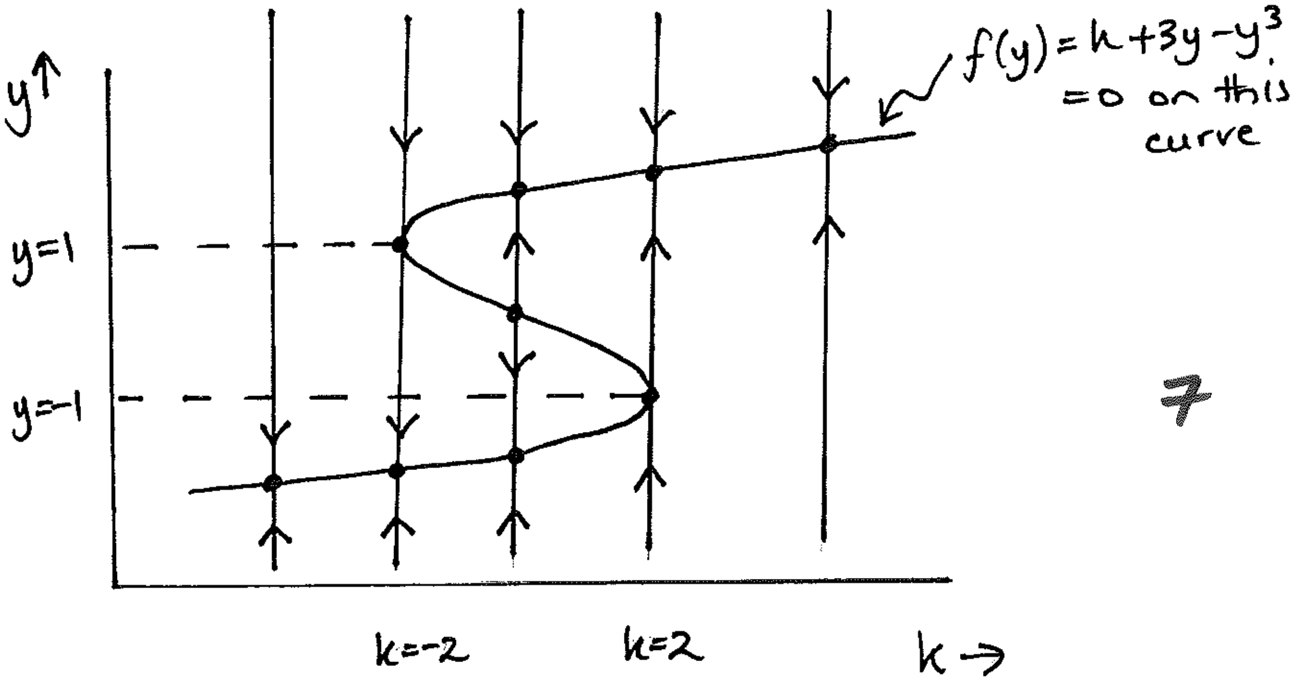

dx/dt = k + 3 x − x3
k = 2.1
x
t

(b) The bifurcation diagram is as shown below.



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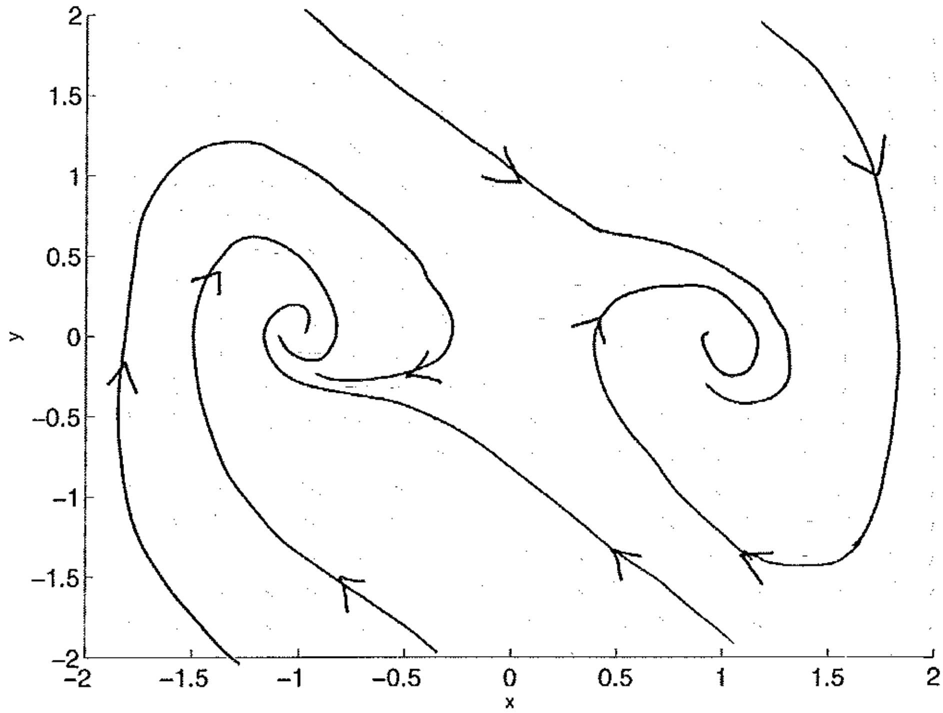
Q5 Total 19

6. The following two pictures show direction fields for the two sets of differential equations given. On each picture, sketch a representative sample of solution curves, making sure to show the direction of time with arrows. (Hand in this page with your solutions).

(a)

$$\dot{x} = y$$
$$\dot{y} = x - y - x^3$$



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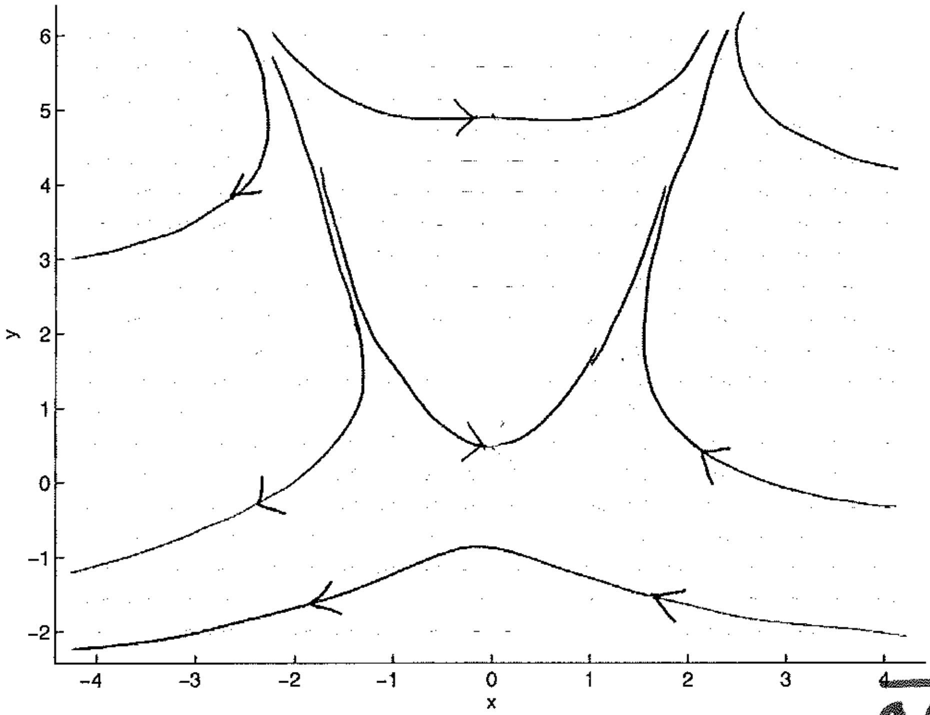
(b)

$$\dot{x} = y - x^2$$
$$\dot{y} = x$$



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Q6 10

Assignment Total 82