Maths 260 Lecture 14

- **Topics for today:**
  Introduction to systems of differential equations
  Vector fields, direction fields and solutions
- **Reading for this lecture:** BDH Section 2.2
- **Suggested exercises:** BDH Section 2.2, #1,3,11,13-16,19,27
- **Reading for next lecture:** BDH Section 2.4
- **Today's handouts**: Assignment 2, Exercises on eigenvalues and eigenvectors, Lecture 14 pictures.

Systems of First Order DEs

- DEs that contain more than one dependent variable are known as systems of DEs.

Example 1:

$$\frac{dx}{dt} = -2x + 3y,$$
$$\frac{dy}{dt} = -2y$$

Example 2:

$$\begin{aligned}
\frac{dx}{dt} &= 10(y - x) \\
\frac{dy}{dt} &= 28x - y - xz \\
\frac{dz}{dt} &= -\frac{8}{3}z + xy
\end{aligned}$$

Example 3:

$$\begin{aligned}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= x - x^3 - y + \mu \cos(t)
\end{aligned}$$

Standard form

- We are mostly interested in systems of first order DEs. We write these in standard form:

$$
\begin{aligned}
\frac{dx_1}{dt} &= f_1(t, x_1, x_2, \ldots, x_n) \\
\frac{dx_2}{dt} &= f_2(t, x_1, x_2, \ldots, x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= f_n(t, x_1, x_2, \ldots, x_n)
\end{aligned}
$$

- The notation $\dfrac{dx_1}{dt} \equiv \dot{x}_1, \quad \dfrac{dx_2}{dt} \equiv \dot{x}_2$, is often used.

Solutions to systems of DEs

- A solution to a system of n first order equations is a set of n functions that satisfy the differential equations.

Example 4: Determine which of the following pairs of functions is a solution to the system

$$\frac{dx}{dt} = -2x + 3y,$$
$$\frac{dy}{dt} = -2y.$$

- $x(t) = -3te^{-2t}, \quad y(t) = -e^{-2t};$
- $x(t) = 3e^{-2t}, \quad y(t) = 0;$
- $x(t) = 3e^{-2t} + te^{-2t}, \quad y(t) = -e^{-2t};$

Solutions to systems of DEs

$$\frac{dx}{dt} = -2x + 3y, \qquad \frac{dy}{dt} = -2y.$$

1. $x(t) = -3te^{-2t}, \quad y(t) = -e^{-2t}$

Solutions to systems of DEs

$$\frac{dx}{dt} = -2x + 3y, \qquad \frac{dy}{dt} = -2y.$$

2. $x(t) = 3e^{-2t}, \quad y(t) = 0$

Solutions to systems of DEs

$$\frac{dx}{dt} = -2x + 3y, \qquad \frac{dy}{dt} = -2y.$$

3. $x(t) = 3e^{-2t} + te^{-2t}, \quad y(t) = -e^{-2t}$

Vector notation

- Consider a system of two autonomous DEs:

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

- We write

$$\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \mathbf{V}(\mathbf{Y}) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}.$$

- Then the system written in vector form is

$$\frac{d\mathbf{Y}}{dt} = \mathbf{V}(\mathbf{Y}).$$

- $\mathbf{V}(\mathbf{Y})$ is known as a **vector field**, i.e., it is a function that assigns a vector to each point of the (x, y) plane.

Example 5: Write the following system in vector notation:

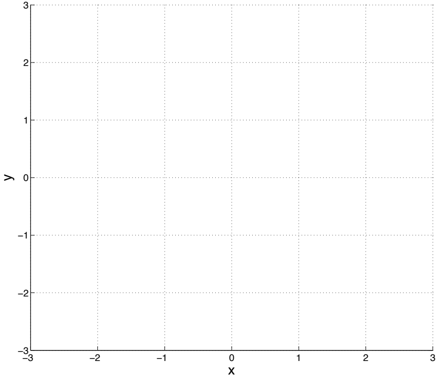
$$\frac{dx}{dt} = 0.5x - 0.4xy$$
$$\frac{dy}{dt} = -y + 0.2xy$$

Drawing vector fields for 2D autonomous systems

- Using vector fields allows us to visualise solutions of systems of DEs, just as slope fields gave us a way to visualise solutions to DEs with one dependent variable.
- 1. For selected points in the $x - y$ plane (say at all points on an evenly spaced grid) calculate $\mathbf{V}(\mathbf{Y})$.
  2. For each point $\mathbf{Y}_0 = (x_0, y_0)$ selected in Step 1, draw the vector $\mathbf{V}(\mathbf{Y}_0)$, with the base of the vector at $\mathbf{Y}_0$ and with an arrow showing the direction of the vector.
- The resulting picture is called a vector field for the DE.
- The Matlab function *pplane* plots vector fields.

Example 6 : Plot the vector field for the following DE.

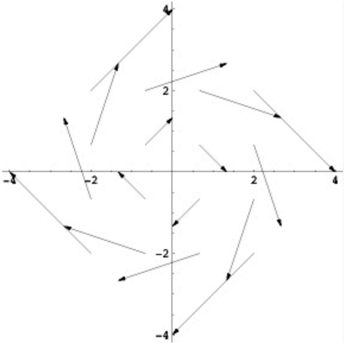
$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x$$



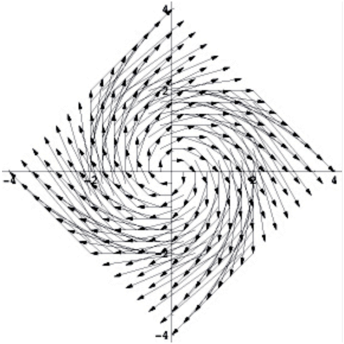
More on vector fields

- One problem with plotting vector fields is that the vectors can cross, which makes a mess. For example, for the previous system:

Vector field with selected vectors

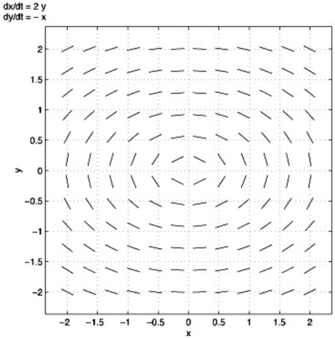


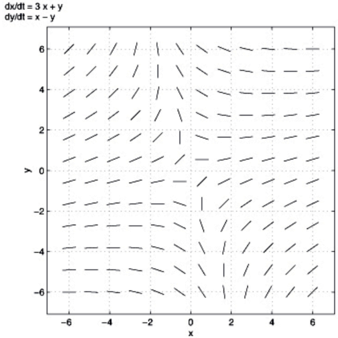
Vector field with more vectors



Direction field

- We can avoid these problems by plotting the **direction field**, i.e., vectors with same direction as in the vector field but scaled to a uniform length.
- Arrows are often omitted. The program *pplane* will plot either the vector field, or the direction field without arrows.
- The following pictures show the direction fields for some systems.





Sketching solutions to systems of DEs

- Consider a system of DEs

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y}), \quad \mathbf{Y} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}. \tag{1}$$

- A solution is a vector of functions $\mathbf{Y}(t)$ and corresponds to a curve in phase space, parameterised by time. That is, as we vary t, we move along the solution curve.

- The vector

$$\left.\frac{d\mathbf{Y}}{dt}\right|_{t=t_0}$$

is tangent to the solution curve for $\mathbf{Y}(t)$ at $t = t_0$.

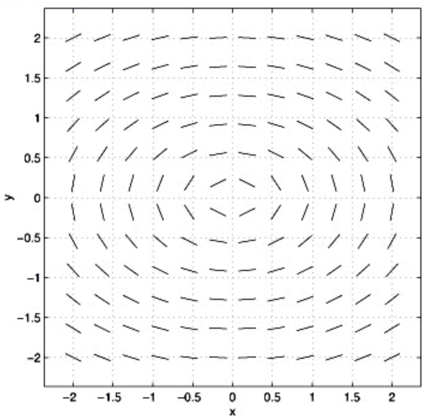
Sketching solutions to systems of DEs

- Thus, equation (1) says that vectors in the direction field are tangent to solutions of the DE.
- So, to sketch solution curves for a DE:
  1. plot the direction field, then
  2. starting at some initial point, sketch a smooth curve that follows the line segments of the direction field.

Example 7: Sketch some solutions for the system

$$\dot{x} = 2y$$
$$\dot{y} = -x$$

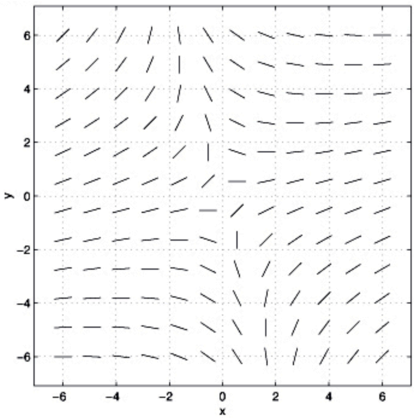
The direction field is given below.



Example 8: Sketch some representative solutions for the system

$$\dot{x} = 3x + y$$
$$\dot{y} = x - y$$

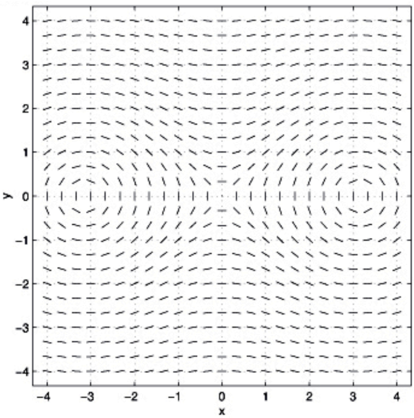
The direction field is given below.



Example 9: Sketch some representative solutions for the system

$$\dot{x} = y$$
$$\dot{y} = \sin x$$

The direction field is given below.



Phase space

- Phase space is the higher dimensional equivalent of the phase line seen in earlier lectures for DEs with one dependent variable. Solutions curves drawn in phase space do not show explicit values of t, just how the dependent variables change as t changes.
- It is common to draw an arrow on a solution curve to indicate the direction that is moved along the solution curve as time increases.
- We can the Matlab programme *pplane* to obtain and plot numerical solutions to the system in phase space. Details of how the numerical methods in *pplane* work are in a later lecture.
- Different solution curves plotted on the same phase space picture give the **phase portrait** of the system.

Important ideas from today's lecture:

- We learnt how to check whether a set of functions is a solution to a given system of DEs.
- We introduced the **phase space**, which is the higher dimensional equivalent of the phase line, to help us visualise our solutions.
- We plotted solution curves on the same phase space picture to give us the **phase portrait**.