Maths 260 Lecture 5

- **Topics for today:**
  Order of numerical methods
  Efficiency of numerical methods
- **Reading for this lecture:** BDH Section 7.2-7.4
- **Suggested exercises:** BDH Section 7.3, #6
- **Reading for next lecture:** BDH Section 1.1 again
- **Today's handouts:** Data used in Lecture 5

Order of Numerical Methods

The **order** of a numerical method measures the change in error of a numerical solution as step size is decreased.

With Euler's method we find that the error is approximately halved if the step size is halved (at least if h is sufficiently small).

For example, Euler's method gives us the following results for the IVP

$$\frac{dy}{dt} = y, \qquad y(0) = 1.$$

No. of Steps	$y\,(1)$	error
1	2.000000	0.718
2	2.250000	0.468
4	2.441406	0.277
8	2.565784	0.152
16	2.637928	0.0804
32	2.676990	0.0413
64	2.687345	0.0209
128	2.707739	0.0105
256	2.712992	0.00529
512	2.715632	0.00265

Looking at the same IVP with Improved Euler yields:

No. of Steps	$y(1)$	error
1	2.500000	0.218
2	2.640625	0.0777
4	2.694856	0.0234
8	2.711841	0.00644
16	2.716594	0.00169
32	2.717850	0.000432
64	2.718173	0.000109
128	2.718254	0.0000274
256	2.718275	0.00000689
512	2.718280	0.00000173

The same IVP with RK4 yields:

No. of Steps	$y(1)$	error
1	2.708333	0.00994
2	2.717346	0.000936
4	2.718210	0.0000719
8	2.718277	0.00000498
16	2.718282	0.000000328
32	2.718282	0.0000000215

We see that, for this DE:

- for a fixed step size we get a smaller error with IE and RK4 than with Euler's method, i.e., IE and RK4 are more accurate methods than Euler's method for this problem;
- (more importantly) for IE and RK4 the error decreases faster as the step size is reduced than with Euler's method.

The first observation above is not true for all DEs, i.e., at a fixed stepsize it is **not** always the case that IE is more accurate than Euler's method and that RK4 is more accurate than IE - it depends on the DE.

On the other hand, the second observation **is** true for most DEs for a small enough stepsize.

We make this second idea more precise by defining the *order* of a numerical method.

Order of a numerical method

Define $E(h)$ to be the error in the approximate solution obtained when we solve an IVP using a particular numerical method with step size h.

If

$$|E(h)| \approx kh^p \qquad \text{as} \qquad h \to 0$$

then p is called the **order** of the numerical method.

Here, k is a constant depending on the IVP and the method.

Estimating the order of a numerical method

For a particular IVP and a particular method of order p, for small h

$$E(h) \approx kh^p$$

for some constant k, so

$$\frac{|E(2h)|}{|E(h)|} \approx \frac{k(2h)^p}{kh^p} = 2^p.$$

Taking logs,

$$\ln\left|\frac{E(2h)}{E(h)}\right| \approx p\ln 2.$$

Thus, for small h

$$p \approx \frac{\ln|E(2h)| - \ln|E(h)|}{\ln 2}.$$

For a particular choice of h, the quantity

$$q = \frac{\ln |E(2h)| - \ln |E(h)|}{\ln 2}$$

is called the **effective order of the method** at step size h.

The effective order q approximates the true order p and the approximation gets more accurate as the step size h decreases.

Example 1: Consider the IVP

$$\frac{dy}{dt} = y, \qquad y(0) = 1.$$

Estimates for $y(1)$ calculated with various methods and step sizes were given earlier.

Use these estimates to calculate the effective order of Euler's method for this IVP at $h = 0.125$.

Also calculate the effective order at this step size for IE and RK4.

General result:

It can be proved that

- Euler's method is of order 1
- Improved Euler is of order 2
- Runge-Kutta 4 is of order 4

This result does not depend on the initial value problem being solved. The result tells us about the behaviour of errors when the stepsize gets very small.

Efficiency of numerical methods

Higher order methods take more work per step than Euler's method. However, they usually require fewer steps and less total work to obtain an accurate answer.

The most efficient method for a particular problem is the one that gives the desired accuracy with the least amount of work.

In estimating the amount of work required to use a certain method, we only count the number of evaluations of f, i.e., of the right hand side of the DE. In comparison, additions and multiplications required are negligible.

Example 2: For the IVP discussed at the start of this lecture, which of Euler' method, IE and RK4 is most efficient if an accuracy of 1% in the solution at $t = 1$ is required?

The Dormand-Prince method

The Matlab functions *dfield* and *pplane* use the Dormand-Prince numerical method by default.

We do not study this method in detail, but note that it incorporates three improvements over RK4:

- It is a 5th order method.
- A variable step size is used. The algorithm calculates the step size to be used by estimating the error in each step. A large error estimate will cause a smaller step size to be used.
- Some fitting (with splines) is performed to give smoother solutions.

Important ideas from today:

- Order of a numerical method
- Estimating the order of a numerical method
- Efficiency of numerical methods