# Maths 260 Lecture 19

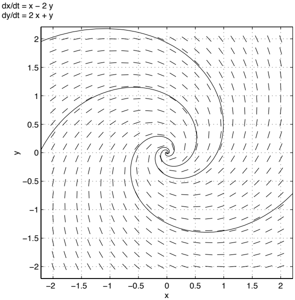
- **Topics for today:**
  Complex numbers
- **Reading for this lecture:**
  Some notes on complex numbers
  BDH Appendix C
- **Reading for next lecture:**
  Some notes on complex numbers
- **Today's handouts**: Tutorial 7

# Example 1

- Consider the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \mathbf{Y}.$$

- Slope field and some solutions:




- There are no straight-line solutions – What is going on?

- Calculate the eigenvalues:

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

**Notes:**

- We need the square root of a negative number!
- The matrix doesn't have any eigenvalues that are **real** numbers. That's why there are no straight-line solutions the system of DEs.
- However, we can still find eigenvalues that are **complex** numbers.
- We can still calculate two eigenvalues provided we introduce a new number:

$$i = \sqrt{-1}$$

# Example 2

- Calculate $\sqrt{-25}$ and $\sqrt{-\frac{16}{9}}$.

- So for previous example, two eigenvalues are:

# Example 3

Find the solutions to $x^2 + 2x + 2 = 0$.

- We can use the quadratic formula to get two (possibly complex) solutions to any quadratic equation

$$ax^2 + bx + c = 0$$

# Fundamental Theorem of Algebra

- More generally, we have the following result:
- An $n$th degree polynomial

$$a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 = 0$$

  has $n$ solutions. This means that the polynomial can be factorised:

$$\begin{aligned} a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 \\ = a_n (x - x_1)(x - x_2)(x - x_3) \dots (x - x_n) \end{aligned}$$

  where $x_1$, $x_2$, ..., $x_n$ are the roots of the equation.
- Note that some of these $x_i$ may be repeated roots.

# Complex numbers

**Definition:** A complex number is a number of the form

$$z = a + ib$$

where $a$ and $b$ are real and $i^2 = -1$.

- We can think of a complex number as a *pair* of real numbers $(a, b)$. Plotting this pair as a point in the plane gives us a geometric interpretation of complex numbers - the **Argand Diagram**.

# Definitions

For a complex number $z = a + ib$,

- The **real part**, written $\operatorname{Re} z$, is $a$.
- The **imaginary part**, written $\operatorname{Im} z$, is $b$.
- The **complex conjugate** is $\bar{z} = a - ib$. It is very useful since $z + \bar{z}$ and $z\bar{z}$ are real.
- The **modulus** is $|z| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}}$.

**Example 4:** $z = 2 - 3i$, compute:

- $\bar{z}$
- $\text{Re } z$
- $\text{Im } z$

**Example 5:** $z = 1 + 6i$, compute:

- $\bar{z}$
- $\text{Re } z$
- $\text{Im } z$

# Arithmetic of Complex Numbers

**Addition/Subtraction:**
collect real and imaginary terms (add/subtract the real and imaginary parts separately).

**Example 6:** $(2 + 4i) + (3 - 2i)$

**Example 7:** $(-1 - 4i) - (4 + 3i)$

**Multiplication:** Multiply out brackets and collect real and imaginary terms, remembering that $i^2 = -1$.

**Example 8:** $(2 + 4i)(3 - 2i)$

**Example 9:** $(-1 - 4i)(4 + 3i)$

**Example 4 again:** $z = 2 - 3i$, compute:

- $z + \bar{z}$
- $z\bar{z}$
- $|z|$

**Example 5 again:** $z = 1 + 6i$, compute:

- $z + \bar{z}$
- $z\bar{z}$
- $|z|$

**Division:** Multiply top and bottom by complex conjugate of denominator, then collect real and imaginary terms.

**Example 10:** $\dfrac{2+4i}{3+2i}$

**Example 11:** $\dfrac{-1-4i}{4-3i}$

# Polar Form

- Another way of describing a complex number is to use **polar co-ordinates**.
- We write
$$z = a + ib = r(\cos\theta + i\sin\theta)$$
where $a = r\cos\theta,\ b = r\sin\theta,\ r = \sqrt{a^2 + b^2},\ \theta = \tan^{-1}(b/a)$
- We call $r$ the **modulus** of z, denoted $|z|$, and we call $\theta$ the **argument** of z, denoted arg z.

**Example 12:** Convert $z = 1 + i$ into polar form:

**Example 13:** Convert $z = \sqrt{2} - i\sqrt{2}$ into polar form:

**Example 14:** Convert $z = 3(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2})$ into rectangular form:

**Example 15:** Convert $z = 2(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$ into rectangular form:

# Important ideas from today's lecture:

- Arithmetic of complex numbers - addition, subtraction, multiplication, division
- Fundamental theorem of algebra: an $n$th order polynomial always has $n$ roots (if complex roots are counted)
- The Argand diagram
- Polar coordinates