

Maths 260 Lecture 7

- ▶ **Topics for today:**
Existence and uniqueness of solutions
- ▶ **Reading for this lecture:** BDH Section 1.5
- ▶ **Suggested exercises:** BDH Section 1.5, #1,3,5,7,15.
- ▶ **Reading for next lecture:** BDH Section 1.6, pp 76-85
- ▶ **Today's handouts:** Tutorial 3 question sheet

Existence and Uniqueness of solutions

In the theory and examples we have studied so far we have been making two major assumptions: that the DEs we study have solutions and that solutions to IVPs are unique.

On the whole we are safe in making these assumptions. Today we shall see why.

Existence Theorem

Consider an initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

If $f(t, y)$ is a continuous function at $(t, y) = (t_0, y_0)$, then there is a constant $\epsilon > 0$, and a function $y(t)$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$ such that $y(t)$ solves the IVP.

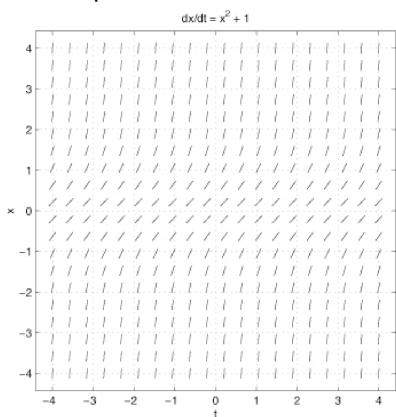
Note: The theorem guarantees a solution exists for a small interval in t , but says nothing about existence for t outside this interval.

Example 1: Consider the IVP

$$\frac{dy}{dt} = 1 + y^2, \quad y(0) = 0.$$

Does the IVP have a solution? If so, for what values of t does the solution exist?

The slope field for the DE is:



Example 2: Does the following IVP have a solution?

$$\frac{dy}{dt} = \frac{1}{t}, \quad y(0) = 1$$

Uniqueness Theorem

Consider an initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

If $f(t, y)$ and $\partial f / \partial y$ are continuous functions at $(t, y) = (t_0, y_0)$, then there is a constant $\epsilon > 0$ and a function $y(t)$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$ such that $y(t)$ is the unique solution to the IVP on this interval.

If the conditions of the Uniqueness Theorem are satisfied, different solutions cannot cross or meet in the (t, y) plane.

Example 3: Does the following IVP have a unique solution?

$$\frac{dy}{dt} = 1 + y^2, \quad y(0) = 0.$$

Example 4: Does the IVP

$$\frac{dy}{dt} = \sqrt{y}, \quad y(2) = 0$$

have a unique solution?

What about the IVP

$$\frac{dy}{dt} = \sqrt{y}, \quad y(0) = 2 ?$$

Example 5: Given the IVP

$$\frac{dy}{dt} = ty^{\frac{1}{5}}, \quad y(t_0) = y_0$$

1. Find a value of t_0 and a value of y_0 so that the IVP has a unique solution. Give a reason for your answer. (You do not try to find an explicit solution to this DE to answer the question.)
2. Find a value of t_0 and a value of y_0 so that the IVP has more than one solution. For your choice of t_0 and y_0 write down two functions that satisfy the DE.

When are the Existence and Uniqueness Theorems useful?

Example 6: Use *dfield* to investigate the qualitative behaviour of solutions to the DE

$$\frac{dy}{dt} = y(y - 4)(2 + \cos^2(y^2))$$

Example 7: For the IVP

$$\frac{dy}{dt} = e^t \sin(y), \quad y(0) = 5$$

use the function *dfield* from Matlab and Euler's method with various step sizes to determine the behaviour of the solution to the DE.

Important ideas from today:

Consider an initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

If f is 'nice', a solution to the IVP exists, at least for t near t_0 .

Also, if f and $\partial f / \partial y$ are 'nice', the solution to the IVP is unique. This implies that solution curves won't cross or touch in (t, y) space.