Department of Mathematics

MATHS 260 Differential Equations
Mid-semester Test, Tuesday, May 5th, 2009

This test contains **FIVE** questions. Attempt **ALL** questions. Show **ALL** your working. You have 50 minutes to do the test. Total marks = 45.

1. (8 marks) Consider the differential equation

$$\frac{dy}{dt} = -\frac{t}{y}.$$

   (a) Find a one-parameter family of solutions to the differential equation.

   (b) Find a solution to the differential equation that also satisfies the initial condition $y(1) = -2$.

   (c) Find the interval of t values for which the solution you found in (b) exists.

2. (16 marks) Consider the one-parameter family of differential equations

$$\frac{dy}{dt} = ay - y^2.$$

   (a) For the case $a = -3$, find all equilibrium solutions and determine their type (e.g., sink, source). Sketch the phase line.

   (b) Repeat (a) for the case $a = 0$.

   (c) Repeat (a) for the case $a = 3$.

   (d) Now let a vary over all possible values. Locate the equilibrium solutions and determine their type. Sketch the bifurcation diagram. Be sure to label the main features of the bifurcation diagram.

3. (9 marks) Consider the following system of differential equations

$$\frac{dY}{dt} = \begin{pmatrix} -2 & 3 \\ 0 & -1 \end{pmatrix} Y,$$

where

$$Y = \begin{pmatrix} x \\ y \end{pmatrix}.$$

(a) Find all straight line solutions to this system of equations.

(b) Find the general solution to this system of equations. Your answer should contain two arbitrary constants.

(c) Find the solution that passes through $(x, y) = (0, 1)$ when $t = 0$. Express your solution in the form $(x(t), y(t))$.

(d) Sketch the phase portrait showing:

- all equilibrium solutions
- all straight line solutions
- the solution curve you found in part (c) above
- three other representative solution curves

(e) Describe what happens to solutions in the long term.

4. (7 marks) The following differential equation has been suggested as a model of the growth of a population of cockroaches in one of the campus restaurants:

$$\frac{dc}{dt} = kc(100 - c) - s.$$

In this equation, c represents the number of cockroaches, measured in thousands (so $c = 1$ means there are 1000 cockroaches in the restaurant) and t is time measured in weeks. The constants $k > 0$ and $s \geq 0$ are parameters in the model.

(a) Briefly say what each term in the model might represent physically, i.e., say what physical phenomenon is being modelled by each term.

(b) If c is very small ($c << 100$) and $s = 0$ then the population of cockroaches is observed to grow at a rate of about 10% per week. What value of k should be used in the model to match this observation?

(c) What is the maximum possible equilibrium population of cockroaches, according to this model?

5. (5 marks) Consider the following differential equation:

$$\frac{dy}{dt} = \frac{y^2}{t} + t.$$

(a) Use the Existence and Uniqueness Theorems to prove that the initial value problem consisting of this differential equation together with the initial condition $y(2) = 0$ has a unique solution.

(b) Write down an initial condition such that the corresponding initial value problem is not guaranteed to have a unique solution, i.e., choose an initial condition so that the Uniqueness Theorem does not apply to the corresponding initial value problem.

Note: You do not need to prove whether or not the initial value problem has a unique solution.