Department of Mathematics

MATHS 260 Differential Equations

Mid-Semester Test

Tuesday, 24 April, 2007

Instructions

- This test contains **SIX** questions. Attempt **ALL** questions.
- The total is **50 marks**.
- Show **ALL** your working.
- You have **60 minutes** to do the test.

1. (8 marks)

   (a) (4 marks) Find the solution to the initial value problem

   $$\frac{dy}{dt} = y^2 \cos t, \quad y(\pi/2) = 1.$$

   (b) (4 marks) Find the general solution to the following differential equation

   $$\frac{dy}{dt} = 5y + e^t.$$

2. (10 marks)
   Consider the following initial value problem

   $$\frac{dy}{dt} = y^2 - t, \quad y(2) = 1.$$

   (a) (2 marks) Does a unique solution of the IVP exist? Give reasons for your answer.

   (b) (6 marks) Use two steps of the Improved Euler method to find an approximation to the solution at $t = 2.4$.

   (c) (2 marks) Suppose you used ten steps instead of two steps as in (b). How would this affect the error in the approximation? Give reasons for your answer. DO NOT CARRY OUT THE TEN STEPS.

3. (10 marks)
   A small population of animals is living in a large game park. Initially they grow at the rate of 5% per year. The maximum population for the resources of the park is 80,000. It is decided to take 500 animals from the park every year.

   (a) (2 marks) Write a differential equation to model P, the population measured in **thousands**.

   (b) (5 marks) Draw a phase line for the differential equation.

   (c) (3 marks) Use the phase line to sketch graphs of P as a function of t when initial population is:

      i. 13,000
      ii. 10,000

4. (10 marks)
Consider following differential equation

$$\frac{dy}{dt} = (y-1)(y-\alpha).$$

   (a) (3 marks) For $\alpha = 2$, find all equilibrium solutions, determine their types (e.g., source, node) and sketch the phase line.

   (b) (7 marks) Draw the bifurcation diagram. Show all your working. Identify any values of α for which a bifurcation exists.

5. (5 marks) For the following system, find all the equilibrium solutions.

$$\begin{aligned}\frac{dx}{dt} &= x + z \\ \frac{dy}{dt} &= x^3 + y \\ \frac{dz}{dt} &= y - z.\end{aligned}$$

6. (7 marks)
Consider the following system of differential equations:

$$\begin{aligned}\frac{dx}{dt} &= y^2 \\ \frac{dy}{dt} &= t - x\end{aligned}$$

with initial conditions $x(0) = 1, y(0) = 2$.

   (a) (2 marks) Show that this IVP has a unique solution.

   (b) (5 marks) Use one step of the Euler method for systems to approximate $x(0.1)$ and $y(0.1)$.