Department of Mathematics

Maths 260 2009S Assignment 1 Answers

August 11, 2009

1. (a) This is a separable equation.

$$\int y\,dy = \int te^{-2t}\,dt \qquad 2$$

$$\Rightarrow \quad \frac{y^2}{2} + c_1 = \frac{te^{-2t}}{-2} - \int \frac{e^{-2t}}{-2}\,dt$$

$$= -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + c_2 \qquad 3$$

Rearranging, we get

$$y^2 = -te^{-2t} - \frac{1}{2}e^{-2t} + k \quad \text{where} \quad k = 2(c_2 - c_1)$$

or

$$y(t) = \pm\sqrt{k - te^{-2t} - \frac{1}{2}e^{-2t}} \qquad 2$$

You can check this is a solution for all choices of k and for both signs of the square root by substituting into the DE.

(b) No. Writing the RHS of the DE as $f(t, y) = g(t)h(y)$ we find $h(y) = 1/y$. We assume $h(y) \neq 0$ when we separate the variables. For this example, $h(y) \neq 0$ for any y and so there can be no missing solutions. 2

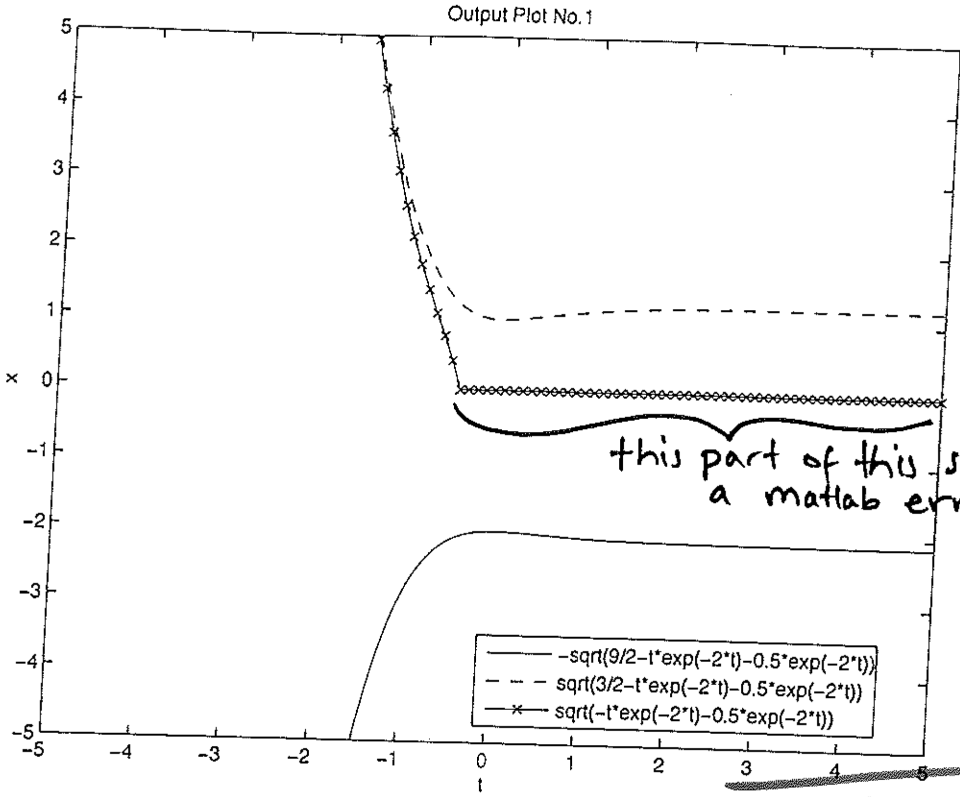
(c) The initial y value is positive, so we must use the positive sign for the square root in the solution to this IVP. Then $y(0) = 1 \Rightarrow 1 = \sqrt{k - 1/2}$ so $k = 3/2$. Thus the solution to the IVP is

$$y(t) = \sqrt{\frac{3}{2} - te^{-2t} - \frac{1}{2}e^{-2t}}. \qquad 2$$

(d) The initial y value is negative, so we must use the negative sign for the square root in the solution to this IVP. Then $y(0) = -2 \Rightarrow -2 = -\sqrt{k - 1/2}$ so $k = 9/2$. Thus the solution to the IVP is

$$y(t) = -\sqrt{\frac{9}{2} - te^{-2t} - \frac{1}{2}e^{-2t}}. \qquad 2$$

(e) The plot on the next page shows the solutions from (c) and (d) and the solution with $k = 0$.

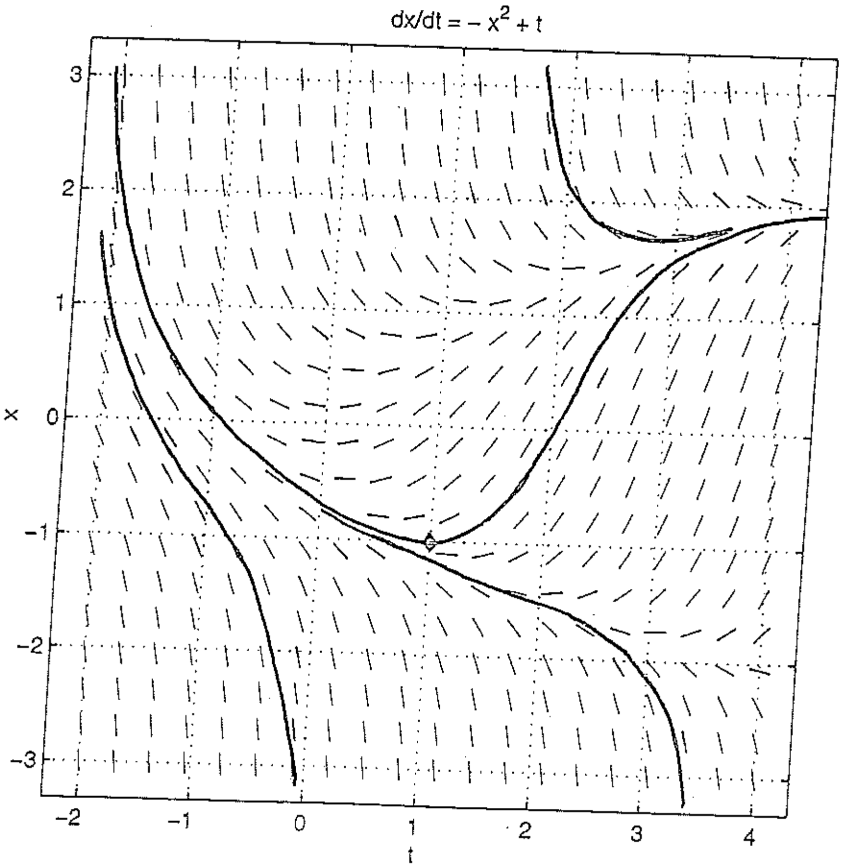


3

this part of this soln is a matlab error.

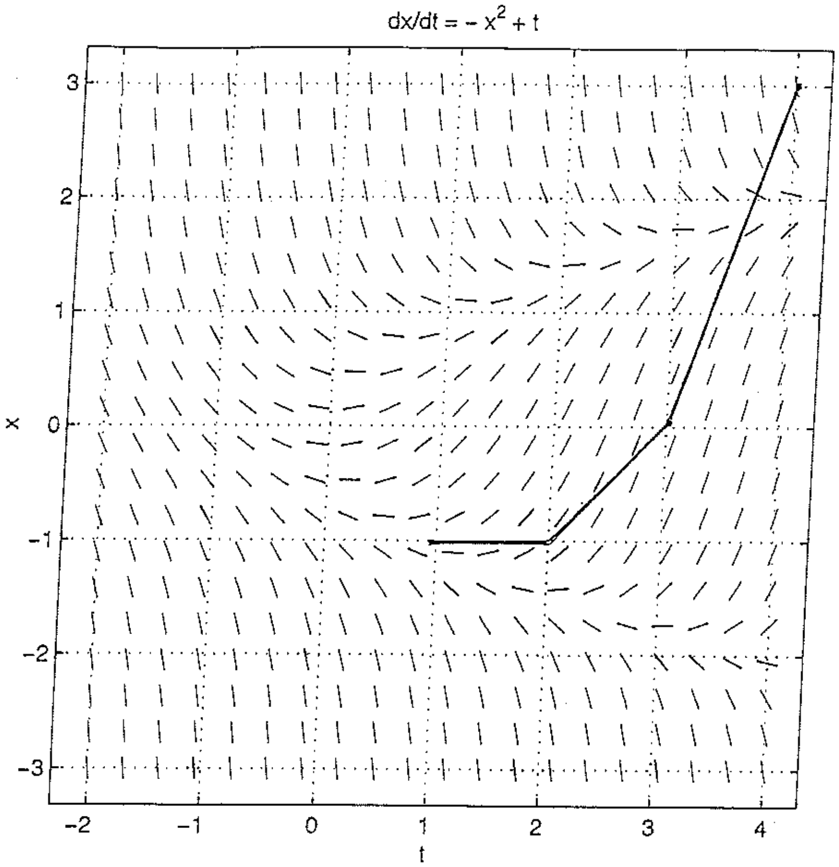
Q1 Total 16

2. (a) Different solution curves should not meet or touch although they will get very close to each other.



(i) 1

(ii) 4



(iii) 3

(b) Using $h = 1$, $t_0 = 1$, $x_0 = -1$, we find the following results.

n	t_n	x_n	$m_1 = f(t_n, x_n)$	$m_2 = f(t_{n+1}, x_n + hm_1)$	$x_{n+1} = x_n + h(m_1 + m_2)/2$
0	1	-1.0	0.0	1.0	-0.5
1	2	-0.5	1.75	1.4375	1.09375
2	3	1.09375	2.80371	-11.19019	-3.09949
3	4	-3.09949			

6

Thus, 3 steps of Improved Euler gives $y(4) \approx -3.09949$. (To get full marks for this question, students are required to show all working needed to calculate the numbers in the table above.)

(c) i. The most accurate approximation to $x(2)$ is probably the estimate obtained with the smallest stepsize. Thus, I estimate $x(2) \approx -0.10289$, accurate to 5 decimal places. 1

ii. The error obtained using 4 steps is approx. $|-0.10372-(-0.10289)| = 0.00083$. The error obtained using 8 steps is approx. $|-0.10295-(-0.10289)| = 0.00006$. 1

iii. A stepsize of $h = 0.125$ means that 8 steps are needed to get from $t = 1$ to $t = 2$. The effective order with this stepsize is

$$\frac{\ln(0.00083) - \ln(0.00006)}{\ln 2} = 3.79.$$

2

iv. The effective order for this method calculated in the previous part of the question is approximately equal to 4. This suggests that a fourth order method was used to get the results. A possible fourth order method is Runge-Kutta 4. 1.

Q2 Total 19

3. (a) Write $f(t,y) = t\sqrt{y}$. Then $f(t,y)$ is defined for $y \geq 0$ and is continuous for all t and for $y > 0$. Also, $\partial f/\partial y = t/(2\sqrt{y})$ which is continuous for all t and for $y > 0$. Since f and $\partial f/\partial y$ are both continuous at the initial condition $(t_0, y_0) = (0, 3)$ the Existence and Uniqueness Theorems tell us that there is a unique solution to the given IVP. 4

(b) Using $y_1(t) = 0$, the left hand side of the DE is

$$\frac{dy_1}{dt} = 0$$

while the right hand side of the DE is 2

$$t\sqrt{y_1} = 0.$$

Since the LHS is equal to the RHS, y_1 is a solution to the DE. Also, $y_1(1) = 0$ so y_1 satisfies the initial condition.

Using $y_2(t) = (t^2 - 1)^2/16$, the left hand side of the DE is

$$\frac{dy_2}{dt} = \frac{t(t^2 - 1)}{4}$$

while the right hand side of the DE is

$$t\sqrt{y_2} = \frac{t\sqrt{(t^2 - 1)^2}}{16} = \frac{t(t^2 - 1)}{4}.$$ 2

Since the LHS is equal to the RHS, y_2 is a solution to the DE. Also, $y_2(1) = 0$ so y_2 satisfies the initial condition.

(c) The functions f and $\partial f/\partial y$ are not continuous at the initial condition $(t_0, y_0) = (1, 0)$ so the hypotheses of the Uniqueness Theorem do not hold and the theorem tells us nothing about the number of solutions to the IVP that exist. Thus there is no contradiction. 2

Q3 Total 10

4. (a) Write $S(t)$ for the amount (in dollars) that Beth has in her account at time t (measured in years). One possible model is

$$\frac{dS}{dt} = 0.03S + 520, \quad S(0) = 400.$$ 3

In writing this model we have modelled the weekly deposits as a continuous rate of deposit and modelled the interest as being continuously compounded. (This model corresponds to a slightly higher rate of interest than 3% but is close enough for our purposes.)

It is also possible to write down a model using time measured in weeks:

$$\frac{dS}{dt} = \frac{0.03S}{52} + 10, \quad S(0) = 400.$$

(b) Using the first model, note that the DE is separable. Separating the variables:

$$\int \frac{dS}{0.03S + 520} = \int dt$$

Making a change in variables $L = 0.03S + 520$ ($\Rightarrow dL = 0.03\ dS$) yields

$$\int \frac{1}{0.03} \frac{dL}{L} = \int dt$$

which, since $L > 0$ can be solved to give

$$L(t) = e^{0.03(t+c)},$$

i.e.,

$$S(t) = \frac{1}{0.03}(ke^{0.03t} - 520)$$ 4

for $k = 0.03e^c$ a positive, arbitrary constant. Using the initial condition $S(0) = 400$ gives $k = 532$, i.e.,

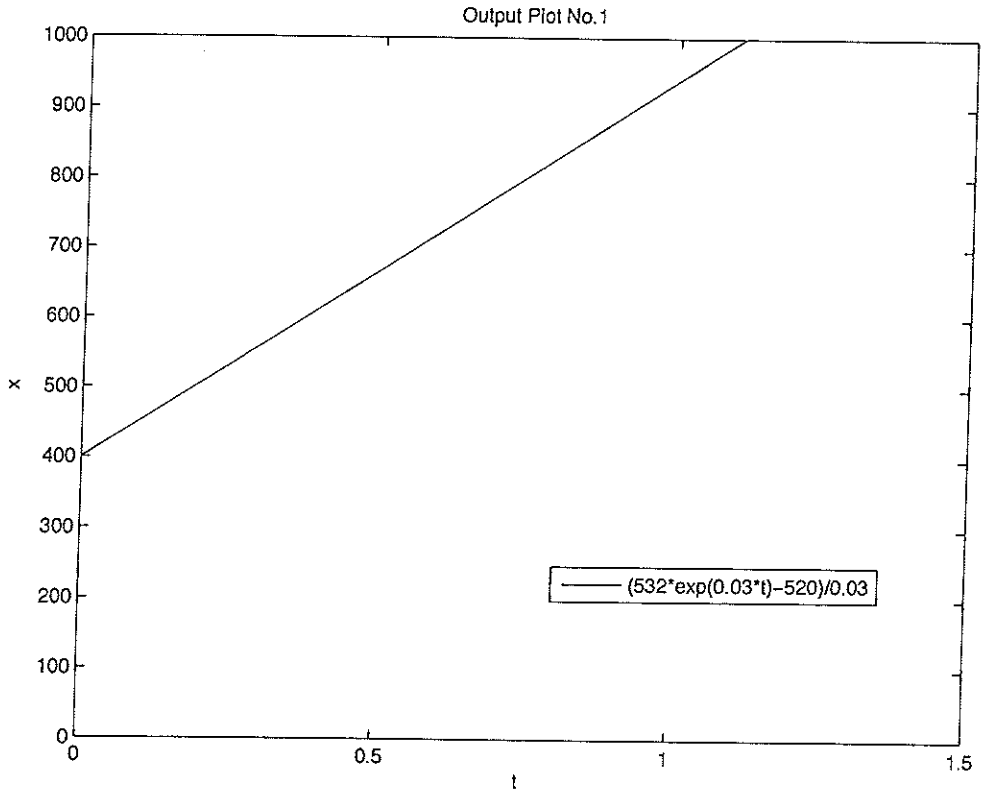
$$S(t) = \frac{1}{0.03}(532e^{0.03t} - 520)$$ 1

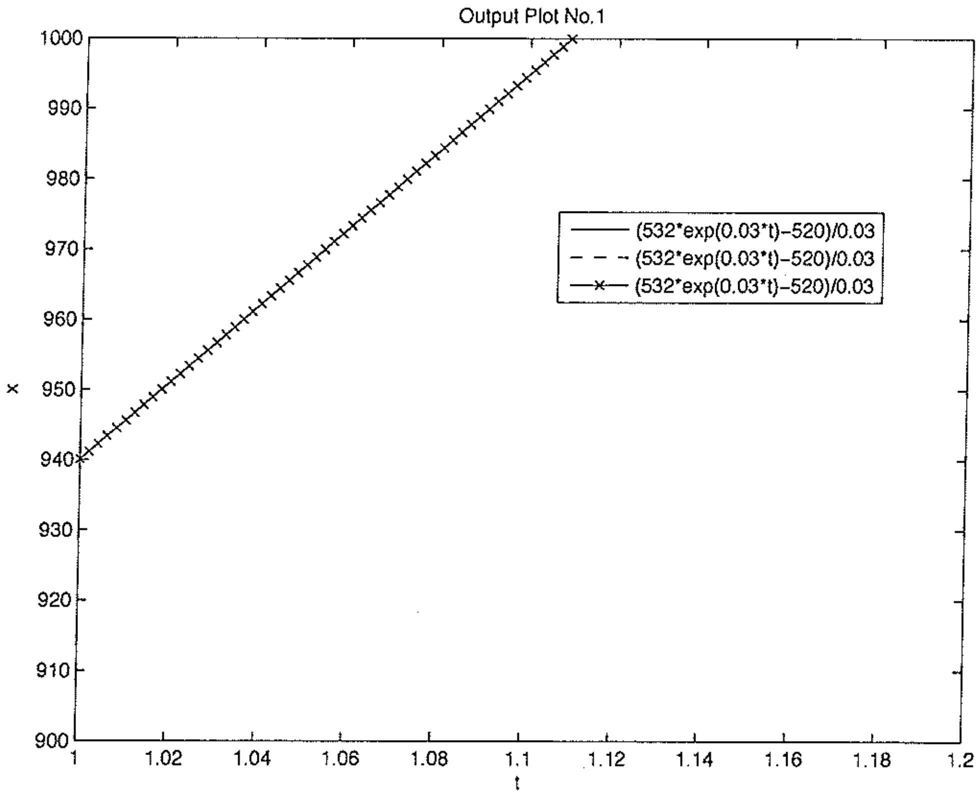
After 4 years,

$$S(t) = \frac{1}{0.03}(532e^{0.03\times 4} - 520) = 2660.94.$$

Thus after 4 years, Beth's account balance is approx. \$2661. 1

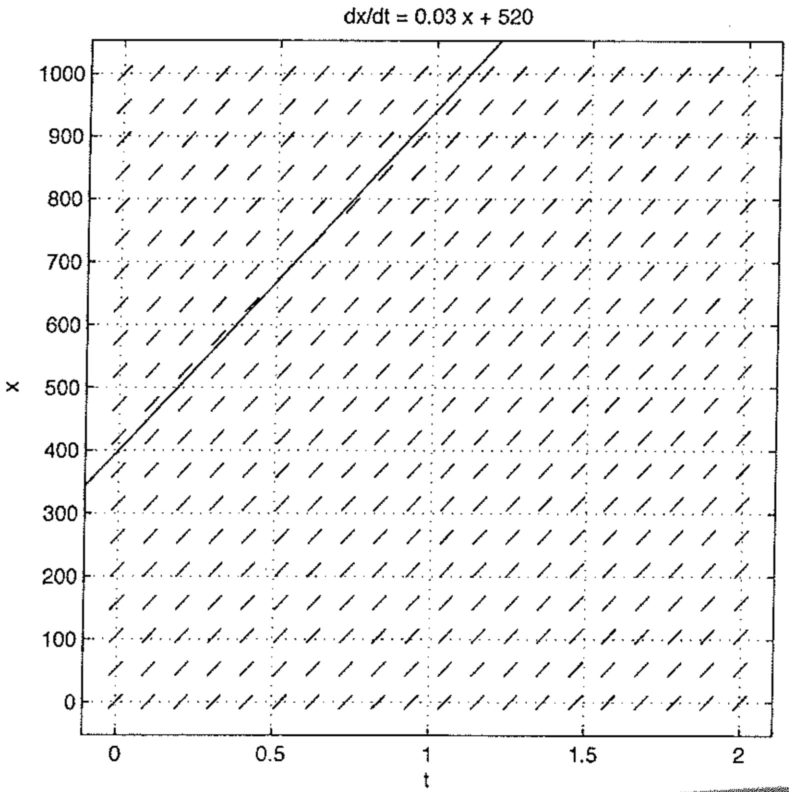
(c) Plotting the function $S(t)$ in analyzer yields the figures below:





From these it is seen that $S(t) = \$1000$ at $t \approx 1.11$, i.e., Beth's balance is \$1000 after 1.11 years, which is about 58 weeks.

Using *dfield* to plot the slope field and entering the initial condition using the keyboard input option gives the following picture, from which it is also seen that the account balance is \$1000 after about 1.1 years.



3 marks

Q 4 Total 12

Assignment Total 57