Mid Semester Test Solns

Semester 1 2005

1. (10 marks)

(a) Find the general solution to the following differential equation

$$\frac{dy}{dt} = t^2 + yt^2,$$

equation is seperable

$$\int \frac{dy}{1+y} = \int t^2\,dt$$

$$\ln|1+y| = \frac{t^3}{3} + c$$

$$1+y = Ae^{t^3/3}$$

$$y = Ae^{t^3/3} - 1$$

(b) Find the general solution to the following differential equation

$$\frac{dy}{dt} = y + t.$$

linear

$$\frac{dy}{dt} - y = t$$

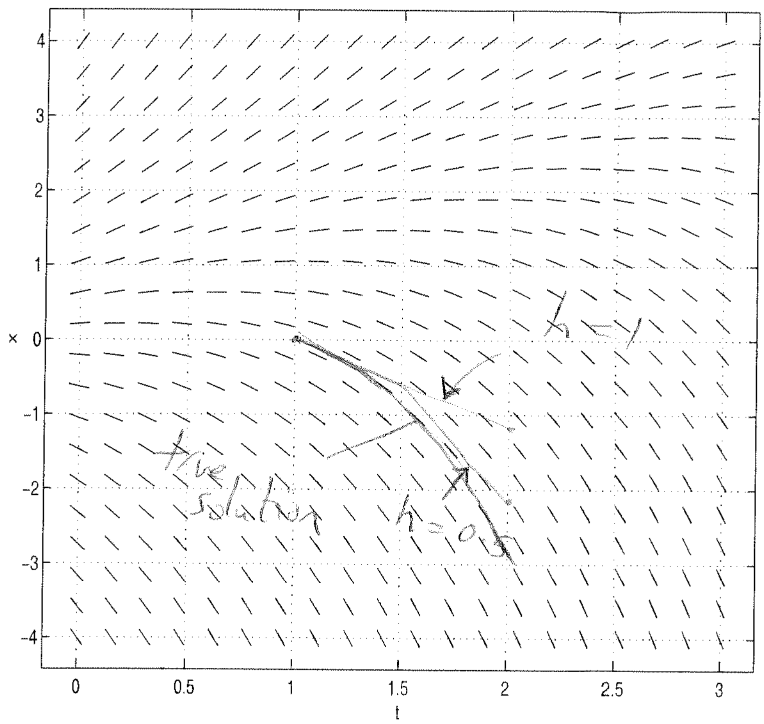
$$\mu = e^{\int -1 \, dt} = e^{-t}$$

$$\frac{d}{dt}\left(y e^{-t} \right) = t e^{-t}$$

$$y e^{-t} = -t e^{-t} - e^{-t} + c$$

$$y = -t - 1 + c e^{+t}$$

2. (5 marks)
The following picture shows the slope field for a differential equation.



(a) On this picture, carefully draw the solution you would obtain if you used one step of Euler's method with $h = 1$ to approximate at $t = 2$ the solution to the differential equation satisfying the initial condition $x(1) = 0$.

(b) On the same picture, carefully draw the solution you would obtain if you used two steps of Euler's method with $h = 0.5$ to approximate the same solution.

(c) Estimate the errors in the approximate solutions you obtained in (a) and (b) at $t = 2$.

(c) True solution $x(2) = -3$

$h = 0.5$ $\quad x(2) = -2.2$

$h = 1$ $\quad x(2) = -1.2$

Error $h = 0.5 = 0.8$

Error $h = 1 = 1.8$

3. (5 marks)
Consider the following initial value problem

$$\frac{dy}{dt} = y\sin(t),\ y(0) = 1.$$

(a) Does a unique solution of the IVP exist? Give reasons for your answer.

(b) Use Improved Euler with stepsize $h = 1$ to find an approximation to the solution at $t = 1$.

(a) We know that $f(y,t) = y \sin t$ is continuous for all y & t & that $\frac{\partial f}{\partial y} = \sin t$ is continuous for all y & t. Therefore, by the Uniqueness theorem there exists a unique soln.

(b) $y(1) \approx y_1 = y_0 + \frac{h}{2}\Big(f(t_0, y_0) + f(t_1, y_0 + h f(t_0, y_0))\Big)$

$= 1 + \frac{1}{2}\big(f(0,1) + f(1, 1 + f(0,1)\big)$

$= 1 + \frac{1}{2}\big(0 + f(1,1)\big)$

$= 1 + \frac{1}{2} \sin 1$

$= 1.4207$

4. (10 marks)

Consider the following differential equation:

$$\frac{dy}{dt} = y^2 + 2y + 1 - \mu$$

(a) Find all equilibrium solutions and determine their types (e.g., sink, node).

(b) Draw the bifurcation diagram. Identify any values of μ for which a bifurcation exists.

(a) $y^2 + 2y + 1 - \mu = 0$

$\Rightarrow y = \dfrac{-2 \pm \sqrt{4 - 4(1-\mu)}}{2} = -1 \pm \sqrt{\mu}$

$\dfrac{df}{dy} = 2y + 2$

At $y = -1 + \sqrt{\mu}$ $\quad \dfrac{df}{dy} = -2 + 2\sqrt{\mu} + 2 = 2\sqrt{\mu}$

at $y = -1 - \sqrt{\mu}$, $\quad \dfrac{df}{dy} = -2\sqrt{\mu}$

Therefore eq. points at $y = -1 \pm \sqrt{\mu}$

for $\mu \geq 0$, $\quad -1 + \sqrt{\mu}$ is source $\mu > 0$

$\quad -1 - \sqrt{\mu}$ is sink $\mu < 0$

& $-1 + \sqrt{\mu} = -1 - \sqrt{\mu}$ is a node for $\mu = 0$

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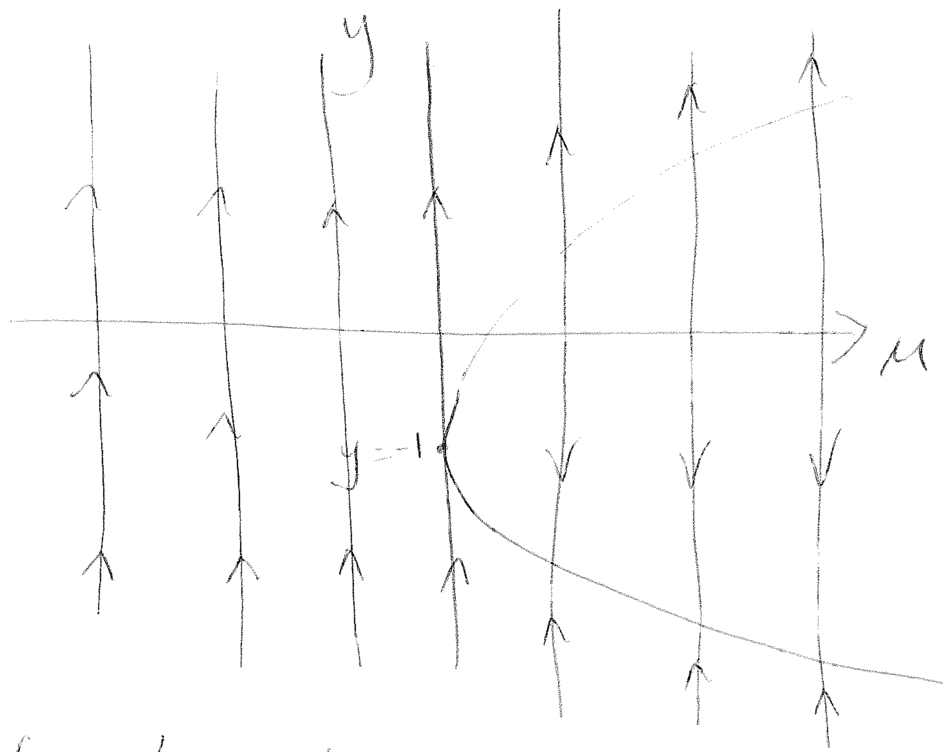
(b)



bifurcation at $\mu = 0$

5. (10 marks)

Consider the following system of differential equations:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(a) Find a solution to this system of equations. Your answer should contain two arbitrary constants.

(b) Find the solution that passes through $(x, y) = (1, 0)$ at $t = 0$. Express your solution in the form $(x(t), y(t))$.

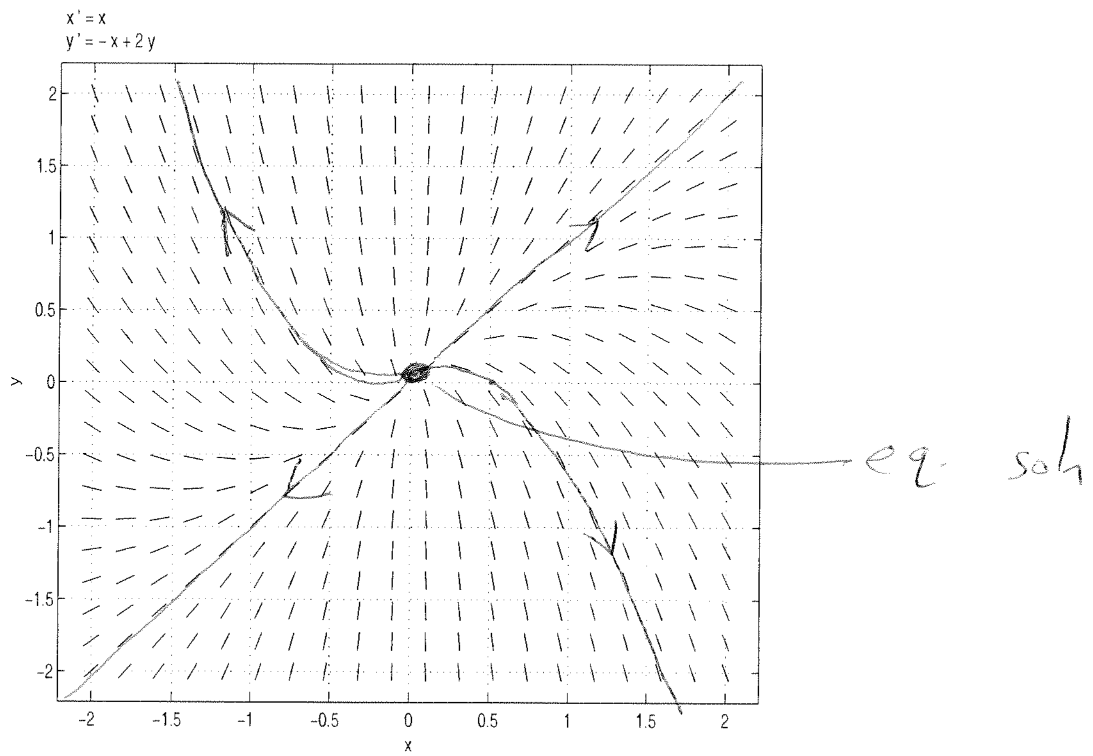
(c) The picture below shows the slope field for the system of equations. On this picture:

i. show all equilibrium solutions;

ii. draw the solution you found in part (b) above;

iii. sketch three other solution curves.

For each solution curve you draw, you should draw an arrow indicating the direction moved as time increases.



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(a) $\frac{dx}{dt} = x \Rightarrow x = c_1 e^t$

$\frac{dy}{dt} = -x + 2y$

$= -c_1 e^t + 2y$

$\Rightarrow \frac{d}{dt}(e^{-2t} y) = -c_1 e^{-t}$

$y = c_1 e^t + c_2 e^{2t}$

(b) $x(0) = 1 \Rightarrow c_1 = 0$

$y(0) = 0 \Rightarrow c_2 = -1$

$x(t) = e^t$

$y(t) = e^t - e^{2t}$