Maths 260SC 2006 Test Solns

① (a) $\frac{dy}{dt} = 3t - \frac{y}{t}, \quad y(1) = 2$

Linear

$$\frac{dy}{dt} + \frac{1}{t} \cdot y = 3t$$

$$\mu(t) = \exp\left(\int \frac{1}{t}\, dt\right)$$
$$= \exp(\ln|t|)$$
$$= |t|$$

$t > 0$ since $\frac{1}{t}$ undefined at $t=0$ and initial t is 1.

$\boxed{\text{Use } \mu(t) = t}$

$$t\frac{dy}{dt} + y = 3t^2$$
$$\frac{d}{dt}(ty) = 3t^2$$
$$ty = t^3 + c$$
$$\boxed{y = t^2 + \frac{c}{t}}$$
$$2 = y(1) = 1 + c$$
$$\Rightarrow c = 1$$
$$\therefore \boxed{y(t) = t^2 + \frac{1}{t}}$$

① (b) $\frac{dy}{dt} = e^{2t}\sqrt{y}$

Separable

$$\int \frac{dy}{\sqrt{y}} = \int e^{2t}\,dt \quad \text{for } y \neq 0$$

$$2\sqrt{y} = \frac{1}{2}e^{2t} + c$$

$$\sqrt{y} = \frac{1}{4}e^{2t} + \frac{c}{2}$$

$$\boxed{y = \left(\frac{1}{4}e^{2t} + k\right)^2} \quad \text{where } k = \frac{c}{2}.$$

$y=0$ is also a solution.

② (a) $f(t,y) = y + 2t$.

This is cts for all t, y.

$\frac{\partial f}{\partial y} = 1$ which is cts for all $t + y$.

$\therefore$ a unique soln exists everywhere + $\therefore$ at $t=0, y=1$.

(b) $h = 0.5,\ t_0 = 0,\ y_0 = 1,$

$\therefore$ require 2 steps so $t_2 = 1$.

Step 1

$$m_1 = f(t_0, y_0) = f(0,1) = \underline{1}$$

$$m_2 = f(t_0 + h, y_0 + hm_1)$$

$$= f(0.5, 1.5)$$

$$= \underline{2.5}$$

$$y_1 = y_0 + \frac{h}{2}(m_1 + m_2)$$

$$= 1 + \frac{0.5}{2}(1 + 2.5) = \underline{\underline{1.875}}$$

② (b) ctd

Step 2

$$m_1 = f(t_1, y_1) = f(0.5, 1.875)$$
$$= \underline{2.875}$$

$$m_2 = f(t_1 + h, y_1 + hm_1)$$
$$= f(1, 1.875 + 0.5 \times 2.875)$$
$$= f(1, 3.3125)$$
$$= \underline{5.3125}$$

$$y_2 = y_1 + \frac{h}{2}(m_1 + m_2)$$
$$= 1.875 + \frac{0.5}{2}(2.875 + 5.3125)$$
$$= \underline{\underline{3.921875}}$$

③ (a) If the Runge-Kutta method is of order k then

$$E(h) \approx Ch^k$$

$$\therefore E\left(\frac{h}{2}\right) \approx C\left(\frac{h}{2}\right)^k = \frac{Ch^k}{2^k} = \frac{E(h)}{2^k}$$

From table results given

$$\frac{E(.05)}{E(.025)} = \frac{1.1332 \times 10^{-5}}{7.3830 \times 10^{-7}} \approx 15.35$$
$$\approx 2^k$$

$\therefore$ expect $k = 4$ since $2^4 = 16$.

alternatively use the effective order formula ($q = 3.94$).

(b) As above $E(h) \approx Ch^4$

$$E\left(\frac{h}{2}\right) \approx \frac{E(h)}{2^4}$$

$$\therefore \text{expect } E(.0125) \approx \frac{E(.025)}{16} = 4.6144 \times 10^{-8}$$

④ $\frac{dy}{dt} = y(y^2 - \mu + 1)$

(a) Em solns when

$y(y^2 - \mu + 1) = 0$

$y = 0$ or $y = \pm\sqrt{\mu - 1}$

$f(y) = y^3 - (\mu-1)y$

$$\boxed{\frac{\partial f}{\partial y} = 3y^2 - (\mu - 1)}$$

$\left.\frac{\partial f}{\partial y}\right|_{y=0} = -(\mu-1)$ >0 for $\mu<1$ source

<0 for $\mu>1$ sink.

$\left.f(y)\right|_{\mu=1} = y^3$ >0 for $y>0$

<0 for $y<0$.

$\left.\frac{\partial f}{\partial y}\right|_{\pm\sqrt{\mu-1}} = 3(\mu-1) - (\mu-1)$

$= 2(\mu-1)$ >0 for $\mu>1$ source

<0 for $\mu<1$ sink.

$y = 0$

$\mu < 1$ source

$\mu = 1$ source

$\mu > 1$ sink.

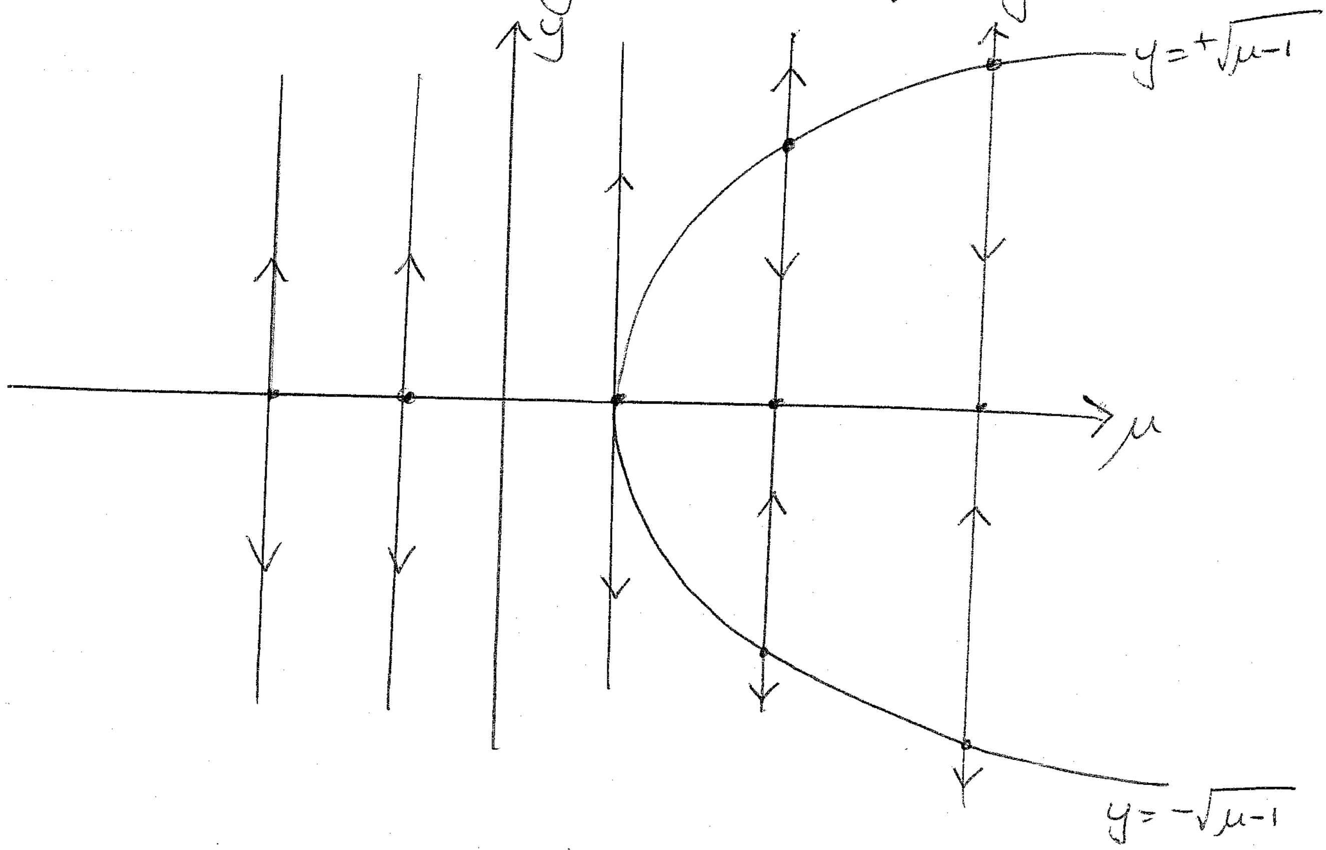
$y = \pm\sqrt{\mu-1}$

$\mu < 1$ sink

$\mu = 1$ source

$\mu > 1$ source.

(4)(b) Em solns when $y=0$ and $\mu = y^2+1$



(5) $$\frac{dY}{dt} = \begin{pmatrix} -3 & 1 \\ 0 & -2 \end{pmatrix} Y$$

(a) Eigenvalues are $-3, -2$ (upper Δr matrix)

Eigenvectors

$\lambda = -3$ $\quad (A - \lambda I) v = 0$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow v_2 = 0 \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\lambda = -2$

$$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow -v_1 + v_2 = 0 \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boxed{Y(t) = c_1 e^{-3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

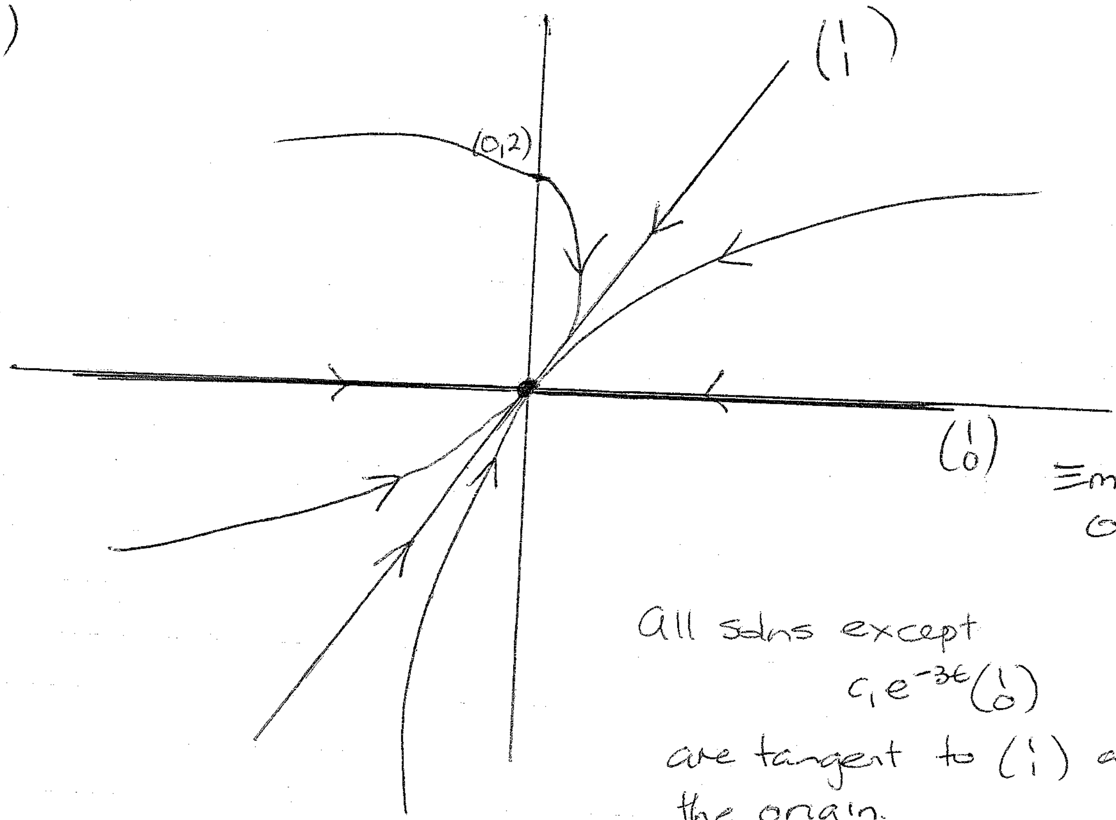
(b)

$$x(t) = c_1 e^{-3t} + c_2 e^{-2t}$$
$$y(t) = c_2 e^{-2t}.$$

$$x(0) = 0 = c_1 + c_2$$
$$y(0) = 2 = c_2 \quad \Rightarrow \boxed{c_2 = 2,\ c_1 = -2}$$

$$x(t) = -2e^{-3t} + 2e^{-2t}$$
$$y(t) = 2e^{-2t}$$

(c)



Eqm soln at origin ●

All solns except
$c_1 e^{-3t}\begin{pmatrix}1\\0\end{pmatrix}$
are tangent to $\begin{pmatrix}1\\1\end{pmatrix}$ at the origin.