THE UNIVERSITY OF AUCKLAND

SECOND SEMESTER, 2006
Campus: City

MATHEMATICS

Differential Equations

(Time allowed: THREE hours)

NOTE: Answer **ALL** questions. Show **ALL** your working. 100 marks in total.

1. (10 marks)

(a) Find the general solution of the differential equation

$$\frac{dy}{dt} = y^2 + 2ty^2.$$

Does your general solution include all possible solutions? Explain your answer.

(b) Solve the initial value problem

$$\frac{dy}{dt} = 2y + e^t, \quad y(0) = 2.$$

2. (12 marks)
Consider the initial value problem

$$\frac{dy}{dt} = t(t + y), \quad y(1) = 2.$$

(a) Use existence and uniqueness theorems to show there exists a unique solution to this initial value problem.

(b) Use two steps of the Improved Euler method to approximate $y(1.4)$.

(c) If you use four steps instead of two steps to approximate $y(1.4)$:

(i) Do you expect the approximation be more accurate?

(ii) What do you expect to happen to the error? Give a reason for your answer.

3. (10 marks) Construct a bifurcation diagram for the differential equation

$$\frac{dy}{dt} = (y - 1)(y - \mu).$$

Be sure to identify any value of μ at which there is a bifurcation. Show all your working.

4. (16 marks) Consider the following two systems of differential equations:

$$(a) \quad \frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{Y} \qquad (b) \quad \frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 2 \\ 0 & -1 \end{pmatrix} \mathbf{Y}$$

For each system:

(i) Determine the general solution. Express your answer in terms of real-valued functions.
(ii) Carefully sketch the phase portrait.
(iii) Describe the long term behaviour of the solutions.

5. (10 marks) Consider the one-parameter family of linear equations

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & a \\ 1 & 1 \end{pmatrix} \mathbf{Y}.$$

Determine the type of equilibrium (e.g. saddle, sink, source) at the origin for all values of a. You do **not** need to sketch any phase portraits.

6. (20 marks) Consider the following system of equations:

$$\frac{dx}{dt} = x(2 - x - y)$$
$$\frac{dy}{dt} = y(3 - 2x - y) \qquad x, y \geq 0.$$

A grid is provided on the yellow answer sheet attached to the back of the question paper. Use the grid for your answer to parts (b) and (c) of this question. Attach the yellow answer sheet to your answer book.

(a) Find all the equilibrium solutions. What does linearisation tell you about the type (e.g. saddle, sink, source) of the equilibrium solutions?
(b) Find the nullclines for the system and sketch them on the answer sheet provided. Show the direction of the vector field in the regions between the nullclines and on the nullclines themselves.
(c) Sketch the phase portrait for the system. Include in your phase portrait the solution curves passing through the initial conditions
   (i) $(x(0), y(0)) = (2, 1)$;
   (ii) $(x(0), y(0)) = (1, 2)$.

   Make sure you show clearly where these solution curves go as $t \to \infty$.

7. (8 marks)
Solve the following initial value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = e^t, \quad y(0) = 0, \ y'(0) = 0.$$

8. (7 marks)
Consider the following differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = 2\cos t.$$

(a) Find the real general solution.

(b) Describe the long term behaviour of the solution.

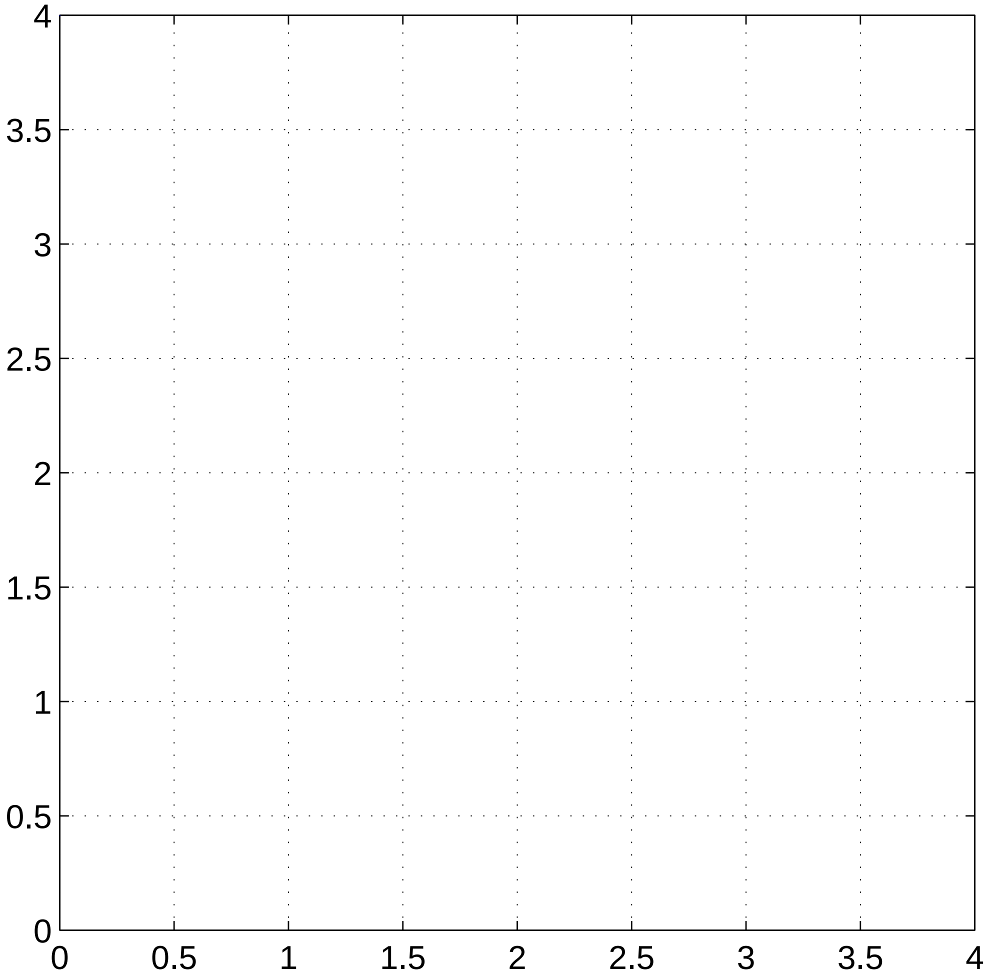
9. (7 marks)
Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = t.$$

Candidate's Name: ______________________ ID No: ____________

TIE THIS ANSWER SHEET TO YOUR ANSWER BOOK

Answer sheet for Question 6



____