Maths 260 Lecture 31

- **Topics for today:**
  Linear, constant coefficient, higher order DEs
  Initial value problems for higher order DEs
  The harmonic oscillator
- **Reading for this lecture:** BDH Section 3.6
- **Suggested exercises:** BDH Section 3.6; 13,15,17,21,23,25
- **Reading for next lecture:** BDH Sections 4.1, 4.2
- **Today's handouts:** None

General result from last lecture:

Consider the differential equation

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0$$

Let $y_1(t), y_2(t), \ldots, y_n(t)$ be n linearly independent solutions of the DE. Then for arbitrary constants c_i,

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

is the **general solution** to the DE. Every solution to the DE can be written in this form by picking the c_i appropriately.

Example 1: Find the general solution to the differential equation

$$2\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 3y = 0$$

Example 2: Find the general solution to the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = 0$$

Example 3: Find the general solution to the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 0$$

Convert to a system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = -4\frac{dy}{dt} - 4y = -4v - 4y$$

So the equivalent system is:

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix} \mathbf{Y}$$

General Method

To find the general solution to

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0$$

- Write down the **characteristic polynomial**:

  $$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0$$

  and find n roots, $\lambda_1, \lambda_2, \ldots, \lambda_n$ (some may be repeated or complex).
- For each λ_i, the function $\mathrm{e}^{\lambda_i t}$ will be a solution to the DE.
- If all the roots are distinct, construct the general solution by taking a linear combination:

  $$y(t) = c_1 \mathrm{e}^{\lambda_1 t} + c_2 \mathrm{e}^{\lambda_2 t} + \cdots + c_n \mathrm{e}^{\lambda_n t}$$

  (converting to real form if necessary).

- If a root (say λ_i) is repeated k times, then the k functions

$$e^{\lambda_i t}, te^{\lambda_i t}, t^2 e^{\lambda_i t}, \ldots, t^{k-1} e^{\lambda_i t},$$

are linearly independent solutions and we can use a linear combination of these in the general solution.

Remember that the general solution to an nth order linear, constant coefficient DE contains n arbitrary constants and n linearly independent solutions.

Example 4: Find the general solution to the differential equation

$$\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 0$$

Example 5: Find the general solution to the differential equation

$$\frac{d^3y}{dt^3} + \frac{dy}{dt} = 0$$

Initial value problems for higher order DEs

Consider a higher order DE such as

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

with associated system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y} = \begin{pmatrix} y \\ v \end{pmatrix}$$

and $v = \frac{dy}{dt}$. To define an IVP for the system we specify an initial condition

$$\mathbf{Y}(t_0) = \mathbf{Y_0} = \begin{pmatrix} y_0 \\ v_0 \end{pmatrix},$$

i.e., $y(t_0) = y_0$ and $v(t_0) = \frac{dy}{dt}(t_0) = v_0$.

The equivalent IVP for the original higher order DE therefore has **two** initial conditions: $y(t_0) = y_0$ and $v(t_0) = \frac{dy}{dt}(t_0) = v_0$.

More generally, an nth order IVP is an nth order DE

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0$$

together with n initial conditions

$$\begin{aligned} y(t_0) &= y_0 \\ \frac{dy}{dt}(t_0) &= y_1 \\ &\vdots \\ \frac{d^{n-1}}{dt^{n-1}}(t_0) &= y_{n-1} \end{aligned}$$

Example 6: Find a solution to the IVP $y'' - 2y' + 10y = 0$, where $y(0) = 0$, $y'(0) = -2$.

Note: here (and elsewhere), $y' = \frac{dy}{dt}, y'' = \frac{d^2y}{dt^2}$.

The Harmonic Oscillator

Consider the second order, linear, constant coefficient DE

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

where $m,\ k > 0,\ b \geq 0$.

Any physical system modelled by this equation is called a **harmonic oscillator**.

For instance, the mass/spring system considered in the last lecture is a harmonic oscillator if we assume linear damping and restoring forces, and no external forcing.

We can now completely classify the different types of solution to this problem.

The characteristic polynomial for the harmonic oscillator is

$$m\lambda^2 + b\lambda + k = 0$$

which has roots

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4mk}}{2m}, \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4mk}}{2m}$$

and the general solution is

$$x(t) = c_1 \mathrm{e}^{\lambda_1 t} + c_2 \mathrm{e}^{\lambda_2 t}$$

There are four different cases, depending on the size of b, the damping coefficient.

Case 1: $b = 0$ (no damping)

Case 2: $0 < b < \sqrt{4km}$ (underdamped)

Case 3: $b > \sqrt{4km}$ (overdamped)

Case 4: $b = \sqrt{4km}$ (critically damped)

Summary

For the harmonic oscillator, modelled by the DE

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

with constants $m,\ k > 0,\ b \geq 0$:

- if $b = 0$ all solutions are periodic except the equilibrium at $x = 0$
- if $b > 0$ all solutions tend to zero as t tends to ∞.

Important ideas from today

To find the general solution to

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = 0$$

- Write down the characteristic polynomial:

  $$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

  and find n roots,$\lambda_1, \lambda_2, \ldots, \lambda_n$. The function $\mathrm{e}^{\lambda_i t}$ will be a solution to the DE.
- If all the roots are distinct, construct the general solution by taking a linear combination:

  $$y(t) = c_1\mathrm{e}^{\lambda_1 t} + c_2\mathrm{e}^{\lambda_2 t} + \cdots + c_n\mathrm{e}^{\lambda_n t}$$

  (converting to real form if necessary).

- If a root (say λ_i) is repeated k times, then the k functions

$$\mathrm{e}^{\lambda_i t}, t\mathrm{e}^{\lambda_i t}, t^2\mathrm{e}^{\lambda_i t}, \ldots, t^n\mathrm{e}^{\lambda_i t},$$

  are linearly independent solutions and we can use a linear combination of these in the general solution.

An nth order IVP is formed from an nth order DE together with n initial conditions.

A **harmonic oscillator** is any physical system modelled by the DE

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

with constants m, $k > 0$, $b \geq 0$:

If $b = 0$ all solutions are periodic except the equilibrium at $x = 0$.
If $b > 0$ all solutions tend to zero as t tends to ∞.